Axioms of independence for structures of preference

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Abstract

Independence is a concept fundamental to non-monotonic reasoning and belief revision. The conditions of trivial independence, symmetry, decomposition, and mixing are a well-established axiomatisation of independence. These are known to be sound and complete for the usual independence relation among random variables on probability distributions. We lift the semantic definitions needed to establish this result to the level of arbitrary structures of preference. As an example of this, we confirm that the same holds for ordinal conditional functions (OCFs): not only are independence statements over OCFs sound with respect to these axioms, but they are also complete. We also show that a faithful operator exists for OCFs, allowing us to deduce that, as is the case for probability spaces, OCFs characterise these axioms.

Keywords

Independence, probability spaces, ordinal conditional functions, preference structures

1. Introduction

Independence among variables has long been a fundamental concern in both non-monotonic reasoning (NMR) and belief revision. Discerning independent subsets among a set of variables aid, for example, in reducing the search space associated to a given query in an inductive system of inference. For many such systems, these problems are known to have high computational complexity. On the other hand, independence is a fundamental characteristic through which one may understand a given knowledge base or belief base, relative to a given system of inference or mode of belief revision. Many such systems are based on structures of preference on sets of possible worlds (be it probability distributions, ordinal conditional functions (OCFs), total preorders, etc.). In this paper we abstract established probabilistic semantics for a syntactic axiomatisation of independence to the level of such structures of preference.

There have been a number of approaches to axiomatising conditions of independence among subsets of a set of variables X. One such approach is the axioms of trivial independence, symmetry, decomposition, and mixing. Geiger, Paz, and Pearl show in [1] that these axioms are both sound and complete for the usual notion of probabilistic independence (where each set of values V for X is assigned a probability distribution). Moreover, via the existence of a faithful operator on probability distributions, it is shown that probabilistic independence precisely characterises those independence relations satisfying the above axioms. [1] also introduces a polynomial-time algorithm that decides whether an independence statement α follows from a set of independence statements A under these axioms. This is in turn used to establish an algorithm to determine whether a set of independence statements A and dependence statements B is consistent with respect to probabilistic independence, i.e. that there exists a probability distribution that satisfies all of A and B.

We observe that many of the definitions and results of [1] can be lifted to the level of an arbitrary structure preference on the set of values V. We introduce language in which to do this. As a special case we study OCFs, which we take to be functions κ with values in $\mathbb{N} \cup \{\infty\}$ for which $\kappa^{-1}(0) \neq \emptyset$. There exist standard notions of independence for OCFs, for example in [2] and [3]. We establish that independence for OCFs is also both sound and complete for the independence axioms. Moreover, we

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show that a faithful operator exists for OCFs, allowing us to deduce that independence for OCFs also characterises the independence axioms, as well as the correctness of the consistency algorithm for OCFs. In doing so, we open the application to OCFs both the practical and the conceptual benefits of the standard axiomatisation for independence. We confirm that a polynomial time algorithm exists to decide whether two sets are independent relative to some unknown OCF, given a set of known independence statements for that OCF. On the other hand we also show that, like for probability spaces, independence among OCFs is precisely a notion of independence among a class of preference structures for which these independence axioms hold.

In Section 2 we recall the definitions of an independence statement α on a set of variables X, an independence relation $\perp\!\!\!\perp$ on X, and the axioms governing independence relations set out in [1]. We also introduce the framework of an abstract class of preference structures $\mathbb{P}$ and the notion of a semantic independence assignment for $\mathbb{P}$. This allows us to lift the definitions of satisfaction, soundness, completeness, and consistency to this level of abstraction. We also lift two results of [1] to this level. In Section 3 we study the semantic independence assignment for probability spaces, and recall the known results of soundness, completeness, and the existence of faithful operators here. In Section 4 we recall some needed definitions associated to OCFs and provide some context as to why the existing notions of independence for OCFs are of interest. In Section 5 we precise the usual semantic independence assignment for OCFs while in Section 6 we establish soundness and completeness for this assignment. Finally, in Section 7, we detail the construction of a faithful operator for OCFs and deduce the corresponding characterisation and consistency results.

2. Axioms for independence

We will be considering independence statements among sets of variables. We fix a finite set of all possible variables $X = \{x_1, \ldots, x_n\}$. Throughout this paper, we allow our variables to take on values from arbitrary finite sets $V_1, \ldots, V_n$. Call the class of all possible value sets $\mathcal{V}$. For a given tuple $(v_1, \ldots, v_n) \in V_1 \times \ldots \times V_n$, we will refer to the assignment $x_1 = v_1, \ldots, x_n = v_n$ as a *state* of X. We will identify a state with the tuple of values $(v_1, \ldots, v_n)$ (implicitly assuming an order on the variables of X). In the special case where each x_i is binary, we have that a state corresponds to the notion of a *world*. In this case, we use the literals x_i and $\overline{x_i}$ to denote an assignment of x_i to 1 and 0, respectively, and we denote a world by a complete conjunction of such literals (with the symbol for $\wedge$ suppressed). So, for example, the state $(0, 1, 1, 0)$ corresponds to the world $\omega = \overline{x_1}x_2x_3\overline{x_4}$. We denote the set of all worlds of X by $\Omega(X)$.

We will study independence in terms of binary relations $\perp\!\!\!\perp \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ on the power set of the set of variables X. A pair $(R, S) \in \mathcal{P}(X) \times \mathcal{P}(X)$ we will refer to as an *independence statement* if R and S are disjoint and we will use the variable α to refer to an arbitrary such statement. If $\perp\!\!\!\perp$ consists solely of independence statements, we will refer to it as an *independence relation*. To denote that $(R, S) \in \perp\!\!\!\perp$ we will write $R \perp\!\!\!\perp S$. In the event that $R = \{x\}$ is a singleton we will denote $R \perp\!\!\!\perp S$ by $x \perp\!\!\!\perp S$. We will denote the union $R \cup S$ as RS. So the language for reasoning about independence that we will use consists of all propositional statements we can form from atoms given by independence statements in $\perp\!\!\!\perp$ over X. A given independence relation $\perp\!\!\!\perp$ may satisfy the following *axioms of independence*, where for any disjoint $R, S, T \subseteq X$, the following hold

(I1)	(Trivial independence)	$S \perp\!\!\!\perp \emptyset$
(I2)	(Symmetry)	$R \perp\!\!\!\perp S \implies S \perp\!\!\!\perp R$
(I3)	(Decomposition)	$R \perp\!\!\!\perp ST \implies R \perp\!\!\!\perp S$
(I4)	(Mixing)	$RS \perp\!\!\!\perp T \wedge R \perp\!\!\!\perp S \implies R \perp\!\!\!\perp ST$

(In the above, the implication is part of a meta-language, while the conjunction is not.) Given a set of independence statements $A = \{\alpha_1, \ldots, \alpha_m\}$, we will denote by $\mathrm{cl}(A)$ the independence relation obtained by taking all atoms in the deductive closure of A under (I1)-(I4). We will denote by $A \vdash \alpha$ the fact that the independence statement α follows from A, that is, that $\alpha \in \mathrm{cl}(A)$.

The following example illustrates these definitions.

Example 1. Consider the set of variables $X = \{h, f, b\}$ expressing properties of a given species of plant. Let the variable h denote the height of a plant, and say it can take on the values $\{s, m, t\}$ denoting short, medium, and tall. Let the variables f and b be binary variables that denote whether or not a plant is flowering and whether or not a plant has bark, respectively. The full set of possible states for X is $V = \{s, m, t\} \times \{0, 1\} \times \{0, 1\}$ while a particular state of X could be $(s, 1, 0)$ denoting that a plant is short, is flowering, and does not have bark. A possible independence statement could be $\alpha = (\{h, b\}, \{f\})$, expressing the fact that both the height of a plant and whether or not a plant has bark are independent from whether or not the plant is flowering. The independence relation $\perp\!\!\!\perp = \text{cl}(\{\alpha\})$ generated by α would then include $\{\alpha, (\{h\}, \{f\}), (\{b\}, \{f\})\}$ along with trivial and symmetric statements. Note that it is not the case that $h \perp\!\!\!\perp b$, as we might expect to naturally be the case.

To provide semantics for independence statements, we will make use of structures of preference on the set of possible states $V = V_1 \times \ldots \times V_n$. These indicate which states are more preferable or likely to occur. Two examples of such structures are probability distributions on V and OCFs on V. Each of these classes of structures has an associated notion of independence, which allows us to recover, from a given preference structure, an independence relation on X. Let $\mathbb{P}$ be a class of preference structures and $\mathbb{P}(V)$ be the class of all such preference structures on V. We use the phrase *semantic independence assignment for V in $\mathbb{P}$* to denote any assignment I_V of each $P \in \mathbb{P}(V)$ to an independence relation $\perp\!\!\!\perp_P$ on X. For an independence statement α on X we will denote by $P \models_{I_V} \alpha$ the fact that $\alpha \in \perp\!\!\!\perp_P$ while for a set of independence statements A, we denote by $P \models_{I_V} A$ the fact that $P \models_{I_V} \alpha_i$ for each $\alpha_i \in A$. We will denote by $A \models_{I_V} \alpha$ the fact that for every $P \in \mathbb{P}(V)$, if $P \models_{I_V} A$ then $P \models_{I_V} \alpha$ also. We use the phrase *semantic independence assignment for $\mathbb{P}$* to denote a family $I = (I_V)$ of semantic independence assignments indexed over all possible sets of states V for all possible variable sets. For A and α on a fixed set of variables X, we will use $A \models_I \alpha$ to denote that $A \models_{I_V} \alpha$ for every possible set of states V for X. We will say that I is *sound with respect to (I1)-(I4)* if for any set of variables X, set of independence statements A over X, and independence statement α over X we have that

$$A \vdash \alpha \implies A \models_I \alpha$$

Conversely, we will say that I is *complete with resepect to (I1)-(I4)* if for any X, A, α we have that

$$A \models_I \alpha \implies A \vdash \alpha$$

We will make extensive use of the contrapositive statement of completeness, which states that for any X, A, α, if $\alpha \notin \text{cl}(A)$, then there must exist some set of states V_α and some preference structure $P_\alpha \in \mathbb{P}(V_\alpha)$ such that P_α satisfies all the statements in A, but not α.

[1] details a polynomial-time algorithm that tests whether an independence statement α follows from a set of independence statements A over X via (I1)-(I4). That is, the definition for a procedure Find(A, α) is given and shown to halt, and return true if and only if $A \vdash \alpha$. The complexity for this algorithm is shown to be $O(|A| \cdot n^2)$ where n is the number of distinct variables of X appearing in $A \cup \{\alpha\}$. This algorithm, along with the following construction, proves useful to establish a result regarding consistency as well as classification for (I1)-(I4). While [1] states these for probability distributions, they in fact apply to any preference structure $\mathbb{P}$ and any semantic independence assignment I for $\mathbb{P}$ that is sound w.r.t. (I1)-(I4). The relevant construction is that of a *faithful operator* $\otimes$, which is studied in [4], and can be defined at the level of some I and $\mathbb{P}$ on a fixed set of variables X. The faithful operator $\otimes$ is an assignment

$$\otimes\colon (V_i, P_i)_{i \in I} \quad \mapsto \quad (V, P)$$

of each family of pairs of sets of states V_i and preference structure $P_i \in \mathbb{P}(V_i)$ to a set of states V and preference structure $P \in \mathbb{P}(V)$ with the property that for any independence statement α on X we have

$$P \models_{I_V} \alpha \iff \forall_{i \in I} \left[P_i \models_{I_{V_i}} \alpha \right] \tag{1}$$

We will be focused on cases where I is finite. The existence of an $\otimes$ that operates on all finite arities reduces to the existence of a binary $\otimes$. The first consequence of the existence of a faithful operator is that complete semantic independence assignments characterise (I1)-(I4).

Theorem 2. *(Geiger, Paz, Pearl, 1991) Suppose I is a semantic independence assignment for a class of preference structures $\mathbb{P}$ which possesses a faithful operator $\otimes$, and which is complete with respect to (I1)-(I4). Then for any independence relation $\perp\!\!\!\perp$ satisfying (I1)-(I4) on a set of variables X there exists a set of states V for X and a preference structure $P \in \mathbb{P}(V)$ such that for any independence statement α on X*

$$\alpha \in \perp\!\!\!\perp \iff P \models_{I_V} \alpha$$

The proof of the above theorem considers all statements $\alpha \notin \perp\!\!\!\perp$, of which there are finitely many since X is finite. By the contrapositive statement of completeness, there must exist a structure P_α that satisfies $\perp\!\!\!\perp$ and not α. Applying $\otimes$ over all such P_α yields the desired P.

All the above constructions are used to devise an algorithm to check consistency of a set of independence statements A and a set of dependence statements (negations of independence statements) B over X. We will define $P \models_{I_V} B$ to mean that $P \not\models_{I_V} \beta$ for every dependence statement $\neg\beta \in B$. We say that A and B are *consistent in I* if there exists some set of states V for X and some $P \in \mathbb{P}(V)$ such that $P \models_{I_V} A$ and $P \models_{I_V} B$. The algorithm $\mathrm{Con}(A, B)$ is defined as follows: if $\mathrm{Find}(A, \beta)$ returns false for each $\neg\beta \in B$, then return true; otherwise return false.

Theorem 3. *(Geiger, Paz, Pearl, 1991) Suppose I is a semantic independence assignment for a class of preference structures $\mathbb{P}$ which possesses a faithful operator $\otimes$, and which is complete with respect to (I1)-(I4). Then for any set of independencies A and set of dependencies B we have that $\mathrm{Con}(A, B)$ returns true if and only if A and B are consistent in I.*

The proof for the case where A and B are consistent in I again relies on the contrapositive form of completeness to declare a P_β that satisfies all of A but not β for every dependence statement $\neg\beta \in B$. We then take $\otimes$ of all of these to produce the desired P.

In the following section, we study the special case where $\mathbb{P}$ is the class of probability distributions. In Section 5 we study the case where $\mathbb{P}$ is the class of ordinal conditional functions.

3. Semantic independence for probability distributions

We now make precise the usual semantic independence assignment for probability distributions. Let $\mathbf{Prob}$ be the class of all probability spaces and for a given set of variables $X = \{x_1, \ldots, x_n\}$ which take on values in $V = V_1 \times \ldots \times V_n$, we denote by $\mathbf{Prob}(V)$ the set of all probability distributions on V. The usual semantic independence assignment for $\mathbf{Prob}$ maps a given distribution $P \in \mathbf{Prob}(V)$, to $\perp\!\!\!\perp_P$, the relation that expresses usual probabilistic independence of subsets of variables w.r.t. P. That is, for disjoint sets $R =, S \subseteq X$, let $P|_R$ and $P|_S$ denote the marginalisations of the distribution P to R and S respectively. Then $R \perp\!\!\!\perp_P S$ if and only if the probability distribution $P|_{RS}$ of P marginalised to the union RS is given by the product distribution $P_R P_S$. We continue our example from before.

Example 4. Let X and V be defined as in Example 1. Now consider the distribution P on V given by

$(t,1,1)$	$\mapsto$	0.14	$(t,1,0)$	$\mapsto$	0.06
$(m,1,1)$	$\mapsto$	0.105	$(m,1,0)$	$\mapsto$	0.045
$(s,1,1)$	$\mapsto$	0.063	$(s,1,0)$	$\mapsto$	0.027
$(t,0,1)$	$\mapsto$	0.007	$(t,0,0)$	$\mapsto$	0.003
$(m,0,1)$	$\mapsto$	0.105	$(m,0,0)$	$\mapsto$	0.045
$(s,0,1)$	$\mapsto$	0.28	$(s,0,0)$	$\mapsto$	0.12

It is easy to check that the independence relation $\perp\!\!\!\perp_P$ associated to this distribution is in fact the same as the independence relation given in Example 1. For example, the distribution $P|_{\{h,b\}}$ is given by

$$\begin{array}{rcl}(t,1) & \mapsto & 0.2\\ (m,1) & \mapsto & 0.15\\ (s,1) & \mapsto & 0.09\\ (t,0) & \mapsto & 0.01\\ (m,0) & \mapsto & 0.15\\ (s,0) & \mapsto & 0.4\end{array}$$

whereas the distribution $P|_{\{f\}}$ is given by

$$1 \mapsto 0.7 \qquad\qquad 0 \mapsto 0.3$$

and $\{hb\}\perp\!\!\!\perp_P\{f\}$ since for any state $(v_1,v_2,v_3) \in V$ we have that $P(v_1,v_2,v_3) = P|_{\{h,b\}}(v_1,v_2)P|_{\{f\}}(v_3)$.

Geiger, Paz, and Pearl consider probabilistic semantics for independence statements in [1]. They establish that the usual semantic independence assignment for **Prob** is both sound and complete with respect to (I1)-(I4). For completeness, they use the contrapositive statement, which says that if we have an independence statement α on X that does not follow from a set A, then there exists some set of states of X and a probability distribution P_α on this set of states such that P_α satisfies A but not α. They moreover establish that we can choose this set of states to be binary.

Theorem 5. *(Geiger, Paz, Pearl, 1991) Let X be any set of variables and I be the usual semantic independence assignment for* **Prob**. *Let A be a set of independence statements for X and $\alpha \notin cl(A)$ be an independence statement for X that does not follow from A under (I1)-(I4). Then there exists some distribution $P_\alpha \in \mathbf{Prob}(\Omega(X))$ such that $P_\alpha \models_{I_{\Omega(X)}} cl(A)$ but $P_\alpha \not\models_{I_{\Omega(X)}} \alpha$.*

An explicit construction of such a P_α is given for any independence statement α. For probability spaces to enjoy Theorem 2 and Theorem 3, [1] constructs a faithful operator $\otimes$ for the usual semantic independence assignment for **Prob**. The binary construction is as follows: given state sets V_1 and V_2 for X and distributions $P_1 \in \mathbf{Prob}(V_1)$ and $P_2 \in \mathbf{Prob}(V_2)$, we let $(V_1,P_2)\otimes(V_2,P_2)$ have state set $V_1 \times V_2$ and probability distribution the product distribution P_1P_2. The fact that this indeed is a faithful operator is proved in [5].

4. Ordinal conditional functions

Ordinal conditional functions (OCFs) or *ranking functions* were first introduced in [6], originally intended to provide a structure by which to represent graded beliefs. We take an OCF[1] on a set of states V to be a function $\kappa : V \to \mathbb{N}_0 \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$. This in general gives an indication of preference or typicality among elements of V. Those states v such that $\kappa(v) = 0$ are considered most preferred, and the larger $\kappa(v)$, the less preferred v is. OCFs arise in various subdisciplines of knowledge representation and reasoning. Two related such disciplines are those of belief revision and NMR. In both these situations, OCFs are used to express preference among states in $\Omega(X)$ on a set of binary variables X. In the context of belief revision, an OCF can be used to represent *epistemic state* of an agent, in which the worlds of rank 0 are those that make up the agent's belief set. This idea is developed in [2] and extends what has been formulated in terms of structures of preference given by total preorders [7]. In NMR, OCFs on $\omega(X)$ are used to construct defeasible entailment relations $\mid\!\sim$ on the class of propositional formulas over the variables in X, where $\phi \mid\!\sim \psi$ models the fact that ψ typically follows from ϕ, but there may be

[1]We remark that this terminology deviates from that used originally in [6]. There OCFs have codomain the class of all ordinals whereas *natural conditional functions* have codomain $\mathbb{N}$. Our definition sits between these two (in that there is only one infinite ordinal in the codomain) and is line with contemporary use of the term (e.g. in [2] and [3]).

exceptions. Defeasible entailment relations constructed from ranking functions are particularly well behaved in that they satisfy the postulates of System P of [8] as well as the Rational Monotony postulate [9]. Examples of well-established entailment relations built from ranking functions are Z-inference [10] and c-inference [11].

As is the case with probability distributions, to speak of independence among OCFs, we must define a suitable notion of marginalisation. We state the definition for variables taking values from arbitrary sets. Suppose X is a set of n variables, $V = V_1 \times \ldots \times V_n$ is a set of states for X, and κ is an OCF on V. Let S be any subset of variables in X. We have the subproduct $V_S = \prod_{x_i \in S} V_i$ of V just containing those values for variables in S and the usual product projection function $\pi_S \colon V \to V_S$. For a given state $v = (v_i)_{1 \leq i \leq n} \in V$ we define $v|_S := (v_i)_{x_i \in S}$ to be the substate of v with values for variables in S. We define the *marginal of κ on S*, denoted $\kappa|_S$, to be the OCF on V_S given by

$$\kappa|_S(v|_S) = \min_{w \in \pi_S^{-1}(v|_S)} \kappa(w)$$

That is, for any state $(v_i)_{x_i \in S}$ for the variables in S, we have that $\kappa|_S$ ranks this state equal to the minimum rank of all states in V that have $(v_i)_{x_i \in S}$ as a substate.

Established notions of independence for OCFs are closely related to *syntax splitting*. In the context of belief revision, this goes back to [12], where the *splitting of a language* refers to a partitioning $X_1 \sqcup X_2$ of the underlying variables X of the language. There, Axiom P is introduced, which expresses the fact that given a splitting, revision by a formula in one sublanguage should only depend on the revision operator of that sublanguage. In [2], a stronger, semantic notion of syntax splitting is proposed. Here, a given epistemic state κ on $\Omega(X)$ is said to *ocf-split* over a partition $X_1 \sqcup \ldots \sqcup X_n$ of X if κ is equal to the sum $\kappa|_{X_1} + \ldots \kappa|_{X_n}$ of the marginals on X_i, taken pointwise. The same definition appears in the context of NMR, for which notions of syntax splitting are introduced in [3]. Here, we refer to a splitting $X = X_1 \sqcup X_2$ of some knowledge base over X as κ-independent for some OCF κ on $\Omega(X)$, if $\kappa = \kappa|_{X_1} + \kappa|_{X_2}$. In the following section, we straightforwardly extend these definitions to situations where variables may not necessarily be binary.

5. Semantic independence for OCFs

We now make precise the usual semantic independence assignment for OCFs. Let $\mathbf{OCF}$ be the class of all possible ordinal conditional functions and $\mathbf{OCF}(V)$ be the set of all ordinal conditional functions on a given set of states $V = V_1 \times \ldots \times V_n$ for a set of n variables X. For any $\kappa \in \mathbf{OCF}(V)$, let $\perp\!\!\!\perp_\kappa$ be defined as follows. For any disjoint sets $R, S \subseteq X$, we have that $R \perp\!\!\!\perp_\kappa S$ if and only if the marginalisation $\kappa|_{RS}$ of κ to the union R and S is given by the sum $\kappa|_R + \kappa|_S$. We use our running example to illustrate the construction.

Example 6. Let X and V be defined as in Example 1. Consider the OCF κ on V given by

$(t,1,1)$	$\mapsto$	0	$(t,1,0)$	$\mapsto$	1
$(m,1,1)$	$\mapsto$	0	$(m,1,0)$	$\mapsto$	1
$(s,1,1)$	$\mapsto$	1	$(s,1,0)$	$\mapsto$	2
$(t,0,1)$	$\mapsto$	2	$(t,0,0)$	$\mapsto$	3
$(m,0,1)$	$\mapsto$	0	$(m,0,0)$	$\mapsto$	1
$(s,0,1)$	$\mapsto$	0	$(s,0,0)$	$\mapsto$	1

Again, the independence relation $\perp\!\!\!\perp_\kappa$ is the same as the independence relation given in Example 1 and Example 4. Here, the marginal $\kappa|_{\{h,b\}}$ is given by

$(t,1)$	$\mapsto$	0
$(m,1)$	$\mapsto$	0
$(s,1)$	$\mapsto$	1
$(t,0)$	$\mapsto$	2
$(m,0)$	$\mapsto$	0
$(s,0)$	$\mapsto$	0

while $\kappa|_{\{f\}}$ is given by

$$1 \mapsto 0 \qquad\qquad 0 \mapsto 1$$

With this we see again that $\{hb\} \perp\!\!\!\perp_\kappa \{f\}$ since for any state $(v_1, v_2, v_3) \in V$ we have that $\kappa(v_1, v_2, v_3) = \kappa|_{\{h,b\}}(v_1, v_2) + \kappa|_{\{f\}}(v_3)$.

6. Soundness and completeness for independence of OCFs

We state soundness and completeness results for the usual semantic independence assignment for OCFs. We start with soundness, the proof for which is a straightforward verification. We detail it to illustrate the underlying algebraic (monoidal) properties satisfied by the independence relation for OCFs. This result is also stated in [13].

Proposition 7. *Let I be the usual semantic independence assignment for OCFs. Then I is sound with respect to (I1)-(I4).*

Proof. Let X be a set of n variables and $V = V_1 \times \ldots \times V_n$ be a set of states for X. Let $\kappa \in \mathbf{OCF}(V)$ and let $\perp\!\!\!\perp_\kappa = I_V(\kappa)$. We must show that $\perp\!\!\!\perp_\kappa$ satisfies (I1)-(I4). For trivial independence, we note that $\kappa|_\emptyset$ is just the trivial ranking function on the empty product, i.e., a one-element set where the one element has rank 0. It is then clear that $\kappa|_{S\emptyset} = \kappa|_S + \kappa|_\emptyset$ for any $S \subseteq X$. So trivial independence follows from the existence of a unit for addition of ranking functions. Symmetry for $\perp\!\!\!\perp_\kappa$ follows immediately from symmetry of addition of ranking functions. Decomposition follows from the following computation: supposing $R \perp\!\!\!\perp_\kappa ST$, we then have for any state μ on RS

$$\begin{aligned}
\kappa|_{RS}(\mu) &= \min_{\omega \in \pi_{RS}^{-1}(\mu)} \kappa(\omega) \\
&= \min_{\omega \in \pi_{RS}^{-1}(\mu)} \kappa|_{RST}(\omega|_{RST}) \\
&= \min_{\omega \in \pi_{RS}^{-1}(\mu)} (\kappa|_R(\omega|_R) + \kappa|_{ST}(\omega|_{ST})) \\
&= \min_{\omega \in \pi_{RS}^{-1}(\mu)} \kappa|_R(\omega|_R) + \min_{\omega \in \pi_{RS}^{-1}(\mu)} \kappa|_{ST}(\omega|_{ST}) \\
&= \kappa|_R(\mu|_R) + \min_{\omega \in \pi_S^{-1}(\mu|_S)} \kappa|_{ST}(\omega|_{ST}) \\
&= \kappa|_R(\mu|_R) + \kappa|_S(\mu|_S)
\end{aligned}$$

Finally, mixing follows directly from associativity of addition of ranking functions: supposing $RS \perp\!\!\!\perp_\kappa T$ and $R \perp\!\!\!\perp_\kappa S$ we get $S \perp\!\!\!\perp_\kappa T$ from decomposition and symmetry and so

$$\begin{aligned}
\kappa|_{RST} &= \kappa|_{RS} + \kappa|_T \\
&= (\kappa|_R + \kappa|_S) + \kappa|_T \\
&= \kappa|_R + (\kappa|_S + \kappa|_T) \\
&= \kappa|_R + \kappa|_{ST}
\end{aligned}$$

□

We next embark on establishing completeness for independence of OCFs. As with the case for probability distributions, we will be proving the contrapositive form of the completeness statement by considering some independence statement α not in the deductive closure of some set of independence statements A and constructing an OCF κ_α that satisfies the statements of A and not α. We can define a preorder $\leq$ on the set of all independence statements on X by taking the product of the usual order on $\mathcal{P}(X)$ with itself. So we set $(S', T') \leq (S, T)$ if and only if $S' \subseteq S$ and $T' \subseteq T$. We set $\alpha' < \alpha$ if $\alpha' \leq \alpha$ and $\alpha' \neq \alpha$. It is sufficient to construct a κ_α for those independence statements α that are *minimal for* A, that is, those α such that for any $\alpha' < \alpha$ we have that $\alpha' \in \mathrm{cl}(A)$. We work towards showing that this can be done for any semantic independence assignment that is sound w.r.t. (I1)-(I4). From repeated applications of symmetry and decomposition we get the following.

Lemma 8. *Suppose $\perp\!\!\!\perp$ is an independence relation on a set of variables X satisfying (I2) and (I3). Then $\perp\!\!\!\perp$ is down-closed with respect to $\leq$.*

So if α' is a statement not in an independence relation $\perp\!\!\!\perp$ satisfying (I1)-(I4), then for any α with $\alpha' \leq \alpha$, we have that α is not in $\perp\!\!\!\perp$ as well. This allows us to deduce the following, which in turn tells us that minimal statements are indeed sufficient to establish our completeness result.

Lemma 9. *Suppose I is a semantic independence assignment for a class of preference structures $\mathbb{P}$ that is sound with respect to (I1)-(I4). Suppose X is a set of variables and V is a set of states for X. Suppose α and α' are independence statements on X with $\alpha' \leq \alpha$ and suppose that A is a set of independence statements on X. Then*

$$A \not\models_{I_V} \alpha' \implies A \not\models_{I_V} \alpha$$

The following observation we take out of the proof for completeness given in [1], since it is purely syntactic in that it is about an arbitrary independence relation $\perp\!\!\!\perp$ satisfying (I1)-(I4) and makes no mention of any preference structure $\mathbb{P}$.

Lemma 10. *(Geiger, Paz, Pearl, 1991) Suppose $\perp\!\!\!\perp$ is an independence relation on X satisfying (I1)-(I4). Suppose $(S_1, S_2), (S_3, S_4)$ are independence statements on X such that (S_1, S_2) is minimal for $\perp\!\!\!\perp$ and $S_1S_2 = S_3S_4$. Then (S_3, S_4) is not in $\perp\!\!\!\perp$.*

The proof for the above considers the intersections $S_{13}, S_{14}, S_{23}, S_{24}$ and proceeds by contradiction, assuming that $(S_3, S_4) = (S_{13}S_{23}, S_{14}S_{24})$ is in $\perp\!\!\!\perp$. Using mixing, we "move S_{23} and S_{14} across the $\perp\!\!\!\perp$" to get $S_1 \perp\!\!\!\perp S_2$. With this equipment in place, we are now able to state our completeness result for OCFs. We again show that the structure that satisfies the given set of statements A but not the $\alpha \notin \text{cl}(A)$ is a structure on the set of binary states.

Theorem 11. *Let X be any set of variables and I be the usual semantic independence assignment for* **OCF**. *Let A be a set of independence statements for X and $\alpha \notin cl(A)$ be an independence statement for X that does not follow from A under (I1)-(I4). Then there exists some $\kappa_\alpha \in \mathbf{OCF}(\Omega(X))$ such that $\kappa_\alpha \models_{I_{\Omega(X)}} cl(A)$ but $\kappa_\alpha \not\models_{I_{\Omega(X)}} \alpha$. Thus, I is complete with respect to (I1)-(I4).*

Proof. By Proposition 7 and Lemma 9, we only need to construct κ_α for those α minimal for A. Let our minimal $\alpha = (S_1, S_2)$. For a given state $\omega \in \Omega(X)$ and subset $S \subseteq X$, let $|\omega|_S$ denote the number of occurrences of positive literals in $\omega|_S$, that is, the number of variables x_i in S such that x_i is true in ω. Now set

$$\kappa_\alpha \colon \omega \mapsto \begin{cases} 0 & \text{if } |\omega|_{S_1S_2} \text{ is even} \\ 1 & \text{if } |\omega|_{S_1S_2} \text{ is odd} \end{cases}$$

κ_α has the property that

> for any $S \subseteq X$ where it is not the case that $S_1S_2 \subseteq S$, we have that the marginal $\kappa_\alpha|_S$ is just the zero ranking function which assigns every state to 0. (2)

Indeed, for any such S there is some variable $x \in S_1S_2$ that does not appear in S and so any ω with x positive and ω' identical to ω but with x negative both restrict to the same state on S. So any such state μ is such that $\pi_S^{-1}(\mu)$ contains at least one world with zero rank.

We now confirm that $\kappa_\alpha \not\models_{I_{\Omega(X)}} \alpha$. Note that, since (S_1, S_2) is minimal, we certainly have that S_1S_2 is nonempty. Now pick any variable $x \in S_1S_2$, and let μ be the state on S_1S_2 where x is positive and all other variables are negative. Then $\kappa_\alpha|_{S_1S_2}(\mu) = 1$. But, by (2), we have that both $\kappa_\alpha|_{S_1}(\mu|_{S_1}) = 0$ and $\kappa_\alpha|_{S_2}(\mu|_{S_2}) = 0$, and so $\kappa_\alpha|_{S_1S_2} \neq \kappa_\alpha|_{S_1} + \kappa_\alpha|_{S_2}$.

Next we show that $\kappa_\alpha \models_{I_{\Omega(X)}} A$. For each $(T_3, T_4) \in A$ we split along two cases. The case where $S_1S_2 \not\subseteq T_3T_4$ is given by (2), which tells us that $\kappa_\alpha|_{T_1T_2}$ is zero here and so $\kappa_\alpha|_{T_1}$ and $\kappa_\alpha|_{T_2}$ are also zero here. We show that the case where $S_1S_2 \subseteq T_3T_4$ cannot occur. Suppose for contradiction that $(T_3, T_4) \in \text{cl}(A)$. Since $\text{cl}(A)$ is closed under decomposition, we have that $(S_3, S_4) \in \text{cl}(A)$ as well, where $S_3 \subseteq T_3$, $S_4 \subseteq T_4$ and $S_3S_4 = S_1S_2$. But by Lemma 10, we have that (S_3, S_4) is not in $\text{cl}(A)$. □

7. A faithful operator for OCFs

As is done for probability spaces, we construct a faithful operator $\otimes$ for the usual semantic independence assignment for OCFs. Again, we only define binary $\otimes$, since n-ary $\otimes$ can be defined by repeated application of binary $\otimes$.

Definition 12. Let V_1 and V_2 be state sets for a set of variables X and let $\kappa_1 \in \mathbf{OCF}(V_1)$ and $\kappa_2 \in \mathbf{OCF}(V_2)$. We define $(V_1, \kappa_1) \otimes (V_2, \kappa_2)$ to have state set $V_1 \times V_2$ and ranking function $\kappa_1 + \kappa_2$.

Letting $V_1 = V_{11} \times \ldots \times V_{n1}$ and $V_2 = V_{12} \times \ldots \times V_{n2}$, we note that it is more accurate to define the state set for $(V_1, \kappa_1) \otimes (V_2, \kappa_2)$ as $(V_{11} \times V_{12}) \times \ldots \times (V_{n1} \times V_{n2})$. But this is in bijection with $V_1 \times V_2$ and so we identify the two via this bijection. To show that $\otimes$ above is indeed a faithful operator, we begin by observing that it remains preserved under marginalisation.

Lemma 13. *Suppose S is a subset of variables from X. Then for any state sets V_1, V_2 on X and ranking functions $\kappa_1 \in \mathbf{OCF}(V_1), \kappa_2 \in \mathbf{OCF}(V_2)$ we have that*

$$((V_1, \kappa_1) \otimes (V_2, \kappa_2))|_S = (V_1, \kappa_1)|_S \otimes (V_2, \kappa_2)|_S$$

Proof. We observe that at the level of state sets we have $(V_1 \times V_2)_S = V_{1S} \times V_{2S}$. Now suppose we have some state $(u_1, u_2) \in V_{1S} \times V_{2S}$. Then

$$\begin{aligned}
(\kappa_1 + \kappa_2)|_S(u_1, u_2) &= \min_{(v_1,v_2) \in \pi_S^{-1}(u_1,u_2)} (\kappa_1 + \kappa_2)(v_1, v_2) \\
&= \min_{(v_1,v_2) \in \pi_S^{-1}(u_1,u_2)} (\kappa_1(v_1) + \kappa_2(v_2)) \\
&= \min_{v_1 \in \pi_S^{-1}(u_1)} \kappa_1(v_1) + \min_{v_2 \in \pi_S^{-1}(u_2)} \kappa_2(v_2) \\
&= \kappa_1|_S(u_1) + \kappa_2|_S(u_2)
\end{aligned}$$

□

Proposition 14. *$\otimes$ is a faithful operator for the usual semantic independence assignment I for* **OCF**.

Proof. Let V_1 and V_2 be state sets for a set of variables X and let $\kappa_1 \in \mathbf{OCF}(V_1)$ and $\kappa_2 \in \mathbf{OCF}(V_2)$. Suppose (S, T) is an independence statement on X. We must show that

$$(\kappa_1 + \kappa_2) \models_{I_{V_1 \times V_2}} (S, T) \iff \kappa_1 \models_{I_{V_1}} (S, T) \wedge \kappa_2 \models_{I_{V_2}} (S, T)$$

We prove the backward implication first. Suppose $\kappa_1|_{ST} = \kappa_1|_S + \kappa_1|_T$ and $\kappa_2|_{ST} = \kappa_2|_S + \kappa_2|_T$. Then

$$\begin{aligned}
(\kappa_1 + \kappa_2)|_{ST} &= \kappa_1|_{ST} + \kappa_2|_{ST} \\
&= (\kappa_1|_S + \kappa_1|_T) + (\kappa_2|_S + \kappa_2|_T) \\
&= (\kappa_1|_S + \kappa_2|_S) + (\kappa_1|_T + \kappa_2|_T) \\
&= (\kappa_1 + \kappa_2)|_S + (\kappa_1 + \kappa_2)|_T
\end{aligned}$$

Conversely, assume that $(\kappa_1 + \kappa_2)|_{ST} = (\kappa_1 + \kappa_2)|_S + (\kappa_1 + \kappa_2)|_T$. That is, for any state $(u_1, u_2) \in V_1|_{ST} \times V_2|_{ST}$, we have that

$$\kappa_1|_{ST}(u_1) + \kappa_2|_{ST}(u_2) = (\kappa_1|_S(u_1|_S) + \kappa_2|_S(u_2|_S)) + (\kappa_1|_T(u_1|_T) + \kappa_2|_T(u_2|_T))$$

Since κ_2 is a ranking function, we know that there is some $u_2 \in V_2|_{ST}$ such that $\kappa_2|_{ST}(u_2) = 0$ and so $\kappa_2|_S(u_2|_S) = 0$ and $\kappa_2|_T(u_2|_T) = 0$ as well. This gives us that $\kappa_1|_{ST} = \kappa_1|_S + \kappa_1|_T$. An appropriate lowest ranking u_1 similarly gives us that $\kappa_2|_{ST} = \kappa_2|_S + \kappa_2|_T$. □

We can use the above along with Theorem 11 to deduce our characterisation and consistency results for OCFs.

Corollary 15. *For any independence relation $\perp\!\!\!\perp$ satisfying (I1)-(I4) on a set of variables X there exists a set of states V for X and an OCF $\kappa \in \mathbf{OCF}(V)$ such that for any independence statement α on X*

$$\alpha \in \perp\!\!\!\perp \iff \kappa \models_{I_V} \alpha$$

We note that the above set of states is not necessarily binary.

Corollary 16. *Let I be the usual semantic independence assignment for OCFs. Then for any set of independencies A and set of dependencies B we have that the algorithm Con(A, B) returns true if and only if A and B are consistent in I.*

8. Related Research

The vast majority of research into axiomatisations of independence relations has focused on the more general case of conditional independence. The core axioms are those of (semi-)graphoids, which were first described as properties of probability distributions in [14] and named and explored more abstractly in [15]. These axioms are significant in that they are closely connected to graphical methods for reasoning about independence statements. One associates to a set of conditional independence statements A a directed acyclic graph (also called an *influence diagram*) which has as nodes variables and as edges the causal relationships between these variables determined by the statements in A. The particular case of interest is where the statements of A form a *causal input list*, where there is a total ordering on A and an element of A expresses independence of some set conditioned on a subset of its ancestors. Independence checking here then corresponds to a graphical procedure called d*-separation*, which can be executed in polynomial time. In [16], it is shown that if we restrict to causal input lists, closure under the semi-graphoid axioms is exactly the same as the set of statements verified by d-separation, and that probabilistic conditional independence is sound with respect to d-separation. In [17], completeness with respect to d-separation of probabilistic conditional independence among causal input lists is established. In [6], a notion of conditional independence is given for natural conditional functions and in [18], soundness of this with respect to the graphoid axioms is established as well as a completeness result for causal input lists. [19] shows that no finite axiomatisation of conditional independence exists for which probabilistic conditional independence is complete with respect to the axiomatisation. [20] establishes the same for natural conditional functions.

9. Further Research

A more recent set of axioms governing independence is set out in [21]. There, a category-theoretic framework is used where independence is phrased in terms of a product-like operation on structures, called an *independent product*, which satisfies conditions akin to a symmetric monoid operation (existence of a unit, symmetry, associativity). Both the product of probability distributions and the sum of ranking functions are examples of such independent products. Following the argument of Proposition 7, we can see that any independent product on a category of preference structures on states will give rise to an independence relation that satisfies (I1)-(I4). A next investigation of interest is to explore other types of independent products (of which there are many examples) and determine whether completeness and the results that follow can be established for these as well. To this end, we have structured our proofs in such a way that as little as possible depends on the type of preference structure in question. It is also interesting that the underlying monoidal structure for the faithful operator is the same as the independent product in both the case of probability spaces and OCFs (but for the faithful operator only the state sets feature in the product, while for the independent product the set of variables also features). The extent to which this generalises can also be investigated. These investigations will not only deepen

our understanding of independence among structures of preference, but may, for example, also open the door to build systems of inference on new classes of preference structures where independence behaves as it does for probability spaces.

We wish to answer above questions for those preference structures given by a *ranking measure* [22], where ranks are taken in an ordered abelian group, and which generalise both probability distributions and OCFs. At a higher level of generality, we could consider *quasi-measures* [22] and ask which of these support the same observations. The same exercise may also be undertaken at the even higher level of generality given by *plausibility measures* [23], where the algebraic structure is not required.

We also wish to extend our investigations to axiomatisations for conditional independence. In particular, we would like to apply known results to recent work in conditional syntax splitting [24, 25]. For example, we wish to determine whether the polynomial-time decision algorithms for d-separation are able to increase the speed of checking typical independence statements in the nonmonotonic inference setting.

10. Conclusion

The axioms of trivial independence, symmetry, decomposition, and mixing ((I1)-(I4)) are well-established to model independence of random variables in probability theory. This statement can be made precise by the fact that the usual semantic independence assignment for $\mathbf{Prob}$ is both sound and complete for these axioms, and, moreover, that every independence relation $\perp\!\!\!\perp$ satisfying (I1)-(I4) possesses a distribution satisfying exactly the statements of $\perp\!\!\!\perp$. We have shown that exactly the same holds for ordinal conditional functions. To achieve this, we have lifted semantic notions of independence to the level of abstract preference structures, and have shown that the notions of soundness, completeness, faithful operators, and consistency can all be formulated at this level. Moreover, the above characterisation result for probability spaces and the correctness of an algorithm for consistency checking hold for any complete semantic independence assignment that has a faithful operator, as is the case for OCFs. It remains to be explored which other classes of preferences structures also exhibit these properties.

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Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

References

[1] D. Geiger, J. Pearl, A. Paz, Axioms and algorithms for inferences involving probabilistic independence, Technical Report R-119, UCLA Computer Science Department, Los Angeles, California, 1989. Also in: Information and Computation 91 (1991) 128-141.

[2] G. Kern-Isberner, G. Brewka, Strong syntax splitting for iterated belief revision, in: Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17, 2017, pp. 1131–1137. URL: https://doi.org/10.24963/ijcai.2017/157. doi:10.24963/ijcai.2017/157.

[3] G. Kern-Isberner, C. Beierle, G. Brewka, Syntax Splitting = Relevance + Independence: New Postulates for Nonmonotonic Reasoning From Conditional Belief Bases, in: Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning, 2020, pp. 560–571. URL: https://doi.org/10.24963/kr.2020/56. doi:10.24963/kr.2020/56.

[4] R. Fagin, Horn clauses and database dependencies, Journal of the Association for Computing Machinery 29 (1982) 952–985. URL: https://doi.org/10.1145/322344.322347. doi:10.1145/322344.322347.

[5] D. Geiger, J. Pearl, Logical and algorithmic properties of conditional independence and graphical models, The Annals of Statistic 21 (1993) 2001–2021. URL: https://www.jstor.org/stable/2242326?origin=JSTOR-pdf.

[6] W. Spohn, Ordinal Conditional Functions: A Dynamic Theory of Epistemic States, Springer Netherlands, Dordrecht, 1988, pp. 105–134. URL: https://doi.org/10.1007/978-94-009-2865-7_6. doi:10.1007/978-94-009-2865-7_6.

[7] A. Darwiche, J. Pearl, On the logic of iterated belief revision, Artificial Intelligence 89 (1997) 1–29. URL: https://www.sciencedirect.com/science/article/pii/S0004370296000380. doi:https://doi.org/10.1016/S0004-3702(96)00038-0.

[8] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, Artificial Intelligence 44 (1990) 167–207. URL: https://www.sciencedirect.com/science/article/pii/0004370290901015. doi:https://doi.org/10.1016/0004-3702(90)90101-5.

[9] D. Lehmann, M. Magidor, What does a conditional knowledge base entail?, Artificial Intelligence 55 (1992) 1–60. URL: https://www.sciencedirect.com/science/article/pii/000437029290041U. doi:https://doi.org/10.1016/0004-3702(92)90041-U.

[10] J. Pearl, System z: a natural ordering of defaults with tractable applications to nonmonotonic reasoning, in: R. Parikh (Ed.), Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning About Knowledge, Morgan Kaufmann, 1990, pp. 121–135. URL: https://api.semanticscholar.org/CorpusID:262088802.

[11] G. Kern-Isberner, Conditionals in Nonmonotonic Reasoning and Belief Revision: Considering Conditionals as Agents, Lecture Notes in Artificial Intelligence, Springer, 2001.

[12] R. Parikh, Beliefs, belief revision, and splitting languages, Center for the Study of Language and Information, USA, 1999, p. 266–278.

[13] W. Spohn, The Laws of Belief: Ranking Theory and Its Philosophical Applications, OUP Oxford, 2012.

[14] A. P. Dawid, Conditional independence in statistical theory, Journal of the Royal Statistical Society. Series B (Methodological) 41 (1979) 1–31. URL: http://www.jstor.org/stable/2984718.

[15] J. Pearl, A. Paz, Graphoids: Graph-Based Logic for Reasoning about Relevance Relations, Technical Report R-53-L, UCLA Computer Science Department, Los Angeles, California, 1985. Also in: B. Du Boulay, D. Hogg, L. Steels (eds.), Advances in Artificial Intelligence II, Elsevier, Amsterdam 1987, pp. 357-363.

[16] T. Verma, J. Pearl, Causal networks: Semantics and expressiveness, in: R. D. Shachter, T. S. Levitt, L. N. Kanal, J. F. Lemmer (Eds.), Uncertainty in Artificial Intelligence, volume 9 of *Machine Intelligence and Pattern Recognition*, North-Holland, 1990, pp. 69–76. URL: https://www.sciencedirect.com/science/article/pii/B9780444886507500111. doi:https://doi.org/10.1016/B978-0-444-88650-7.50011-1.

[17] D. Geiger, J. Pearl, On the logic of causal models, in: R. D. Shachter, T. S. Levitt, L. N. Kanal, J. F. Lemmer (Eds.), Uncertainty in Artificial Intelligence, volume 9 of *Machine Intelligence and Pattern Recognition*, North-Holland, 1990, pp. 3–14. URL: https://www.sciencedirect.com/science/article/pii/B9780444886507500068. doi:https://doi.org/10.1016/B978-0-444-88650-7.50006-8.

[18] D. Hunter, Graphoids and natural conditional functions, International Journal of Approximate Reasoning 5 (1991) 489–504. doi:https://doi.org/10.1016/0888-613X(91)90026-I.

[19] M. Studený, Conditional independence relations have no finite complete characterization, in: Transactions of the 11th Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, volume B, 1990, pp. 377–396.

[20] M. Studený, Conditional independence and natural conditional functions, International Journal of Approximate Reasoning 12 (1995) 43–68. URL: https://www.sciencedirect.com/science/article/pii/0888613X9400014T. doi:https://doi.org/10.1016/0888-613X(94)00014-T.

[21] A. Simpson, Category-theoretic structure for independence and conditional independence, volume 336, 2018, pp. 281–297. The Thirty-third Conference on the Mathematical Foundations of Programming Semantics (MFPS XXXIII).

[22] E. Weydert, General belief measures, in: R. L. de Mantaras, D. Poole (Eds.), Uncertainty in Artificial Intelligence, Morgan Kaufmann, San Francisco (CA), 1994, pp. 575–582. URL: https://www.sciencedirect.com/science/article/pii/B978155860332550078X. doi:https://doi.org/10.1016/B978-1-55860-332-5.50078-X.

[23] N. Friedman, J. Y. Halpern, Plausibility measures: a user's guide, in: Proceedings of the Eleventh Conference on Uncertainty in Artificial Intelligence, UAI'95, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1995, p. 175–184.

[24] J. Heyninck, G. Kern-Isberner, T. Meyer, J. Haldimann, C. Beierle, Conditional syntax splitting for non-monotonic inference operators, Proceedings of the AAAI Conference on Artificial Intelligence 37 (2023) 6416–6424. doi:10.1609/aaai.v37i5.25789.

[25] L.-P. Spiegel, J. Haldimann, J. Heyninck, G. Kern-Isberner, C. Beierle, Generalized safe conditional syntax splitting of belief bases, in: J. Kwok (Ed.), Proceedings of IJCAI-25, 2025, pp. 4678–4686. doi:10.24963/ijcai.2025/521.