Modeling Green Risks with Mixture Copula*

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Abstract

This article applies a mixture copula model to the analysis of green financial risk. The model combines a Gaussian bulk component and a parametric tail component through a mixing weight that is estimated from data alongside the copula parameters – allowing bulk and tail co-movement to be governed by different dependence structures simultaneously. Unlike a single-family copula, which imposes one structure uniformly across the joint distribution, the mixture framework separates the modelling of moderate co-movement under normal conditions from the stronger dependence that defines joint extreme events. The construction is theoretically valid: the resulting density is non-negative everywhere, integrates to unity, and preserves uniform marginals, with no hard boundaries or discontinuities.

Six pairs of variables constructed from combinations of Return on Equity, Debt-to-Equity Ratio, Total Environmental Intensity, and Biodiversity exposure are analyzed on a sample of 570 company-level observations. A Gaussian copula models bulk dependence; Frank, Gumbel, and Clayton copulas are tested as tail components. The mixture specification outperforms every single-family alternative for all six pairs on the Akaike Information Criterion.

The estimated mixing weights range from near zero to 0.53, indicating qualitatively different dependence regimes across pairs. At the practical level, the mixture model changes joint tail risk estimates materially relative to single-copula benchmarks. For Debt-to-Equity Ratio paired with Biodiversity, the 99.5% Conditional Value-at-Risk under the mixture exceeds the single-copula estimate by a factor of 14 – the single copula cannot simultaneously fit near-independent bulk behavior and the non-trivial tail co-movement that governs joint extreme events.

Keywords

Copulas, mixture copulas, composite copulas, joint risks, green risks, mathematical models

1. Introduction

With green risks taking a more prominent role as an object of interest in investments and an increasing integration of environmental, social and governance (ESG) factors in decision making, the research at the intersection of finance and ecological factors is not purely theoretical but a practical topic.

Among various approaches in modeling the dependencies among factors a mathematical model of copulas arises naturally. The ability to separate marginal distributions which might exhibit various complicated behavioral features – heavy tails of financial ratios or skewed environmental metrics – and focus only on dependencies between factors, their joint extremes and at what conditions they occur makes copulas especially useful in context of green risks.

There are several popular classes which are mostly utilized, such as Elliptical and Archimedean. Those families have been developed and utilized already in many fields of science. Their simplicity in usage and flexibility in modeling tail movements is important but, in some cases, limited and dependent on the shape of the distribution, which can exhibit different patterns in terms of symmetry in movement in different parts.

Empirical evidence in financial and environmental data frequently shows that variables behave differently in normal market conditions compared to extreme ones – a phenomenon sometimes

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referred to as tail dependence heterogeneity. A model that fits the bulk or the main body of distribution well may misrepresent the tails, and vice versa. This discrepancy has direct consequences for risk metrics such as Value-at-Risk and Conditional Value-at-Risk, which are defined precisely in the tail region.

To address the issue, it is possible to use a combination of copulas which can be obtained via several methods – patchwork and mixture methods. Rather than selecting a single copula family for the entire distribution, the combined framework blends two copulas – a bulk copula governing the central region and a tail copula governing the extreme region. The transition is continuous and differentiable, preserving a valid joint density that can be evaluated analytically.

This article extends the single-copula analysis performed in our previous article [1], which established Frank, Gumbel and Clayton as the best-fitting families for six ESG-financial variable pairs drawn from a sample of 846 company-level observations. The data have already been presented in other studies [2] which investigated the matter using different mathematical apparatus. The present study replaces those single-family fits with mixture copulas – using the Gaussian copula as the bulk component and testing Clayton, Gumbel and Frank as tail alternatives – and evaluates whether the added flexibility of the mixture structure improves model fit and changes risk estimates materially with a mathematical framework based on Sklar's theorem [3].

2. Related Works

The literature on copulas, and composite copulas in particular, combines both general theoretical foundations and more specialized constructions and applications. A direct starting point is the article [4], where patchwork copulas are used for generating unfavorable VaR scenarios, showing how localized dependence modeling can be applied to tail-oriented risk analysis. The foundational framework for copula theory is established in the book [5], which provides rigorous definitions, Sklar's theorem, and key properties necessary for any copula construction, including patchwork approaches. Developing this further, work [6] broadens the fundamental approach with theory of multivariate dependence, introducing parametric families and structural insights essential for understanding both the flexibility and the constraints of copula modeling. From a risk-management perspective, the book [7] formalizes the role of copulas in tail-risk aggregation, VaR, and stress testing, highlighting the importance of dependence structures beyond marginal distributions.

Theoretical developments specific to patchwork copulas are presented in article [8], where convergence properties under regional constructions are studied, addressing how locally defined copulas extend to valid global objects. A broader theoretical consolidation is given in the book of the same author [9], which systematizes copula properties and provides a general mathematical framework within which composite copula constructions can be interpreted. An alternative construction paradigm is explored in the article [10], where gluing copulas are introduced as a method for combining copulas along boundaries, closely related to patchwork but with stricter structural requirements. Extending flexibility further, the work [11] proposes asymmetric copula constructions, enabling non-exchangeable dependence structures that are often implicitly targeted in copula designs as well as another method based on product of copulas.

A more applied treatment of dependence in finance is given in the resource [12], where copulas are positioned as a central tool for modeling heavy tails and joint extreme events. In empirical modeling, the article [13] demonstrates how copulas capture asymmetric dependencies in financial time series, reinforcing the need for flexible constructions such as patchwork in real-world data. The evaluation of such models is addressed in the article [14], which reviews goodness-of-fit methodologies critical for validating complex copula structures and develops a general approach to handling the copula fit including complex copulas for which conditions of existing approaches might fail. On the computational side, the software paper [15] introduces nonparametric kernel estimation of copulas, enabling flexible approximation of dependence without strict parametric

assumptions which further enables a starting point in building composite copula, while the article [16] provides practical tools for constructing and estimating copula models in applied settings. A comprehensive computational framework is further developed in the book [17], focusing on implementation and model selection using R which can be used to a much greater extent on computationally heavy tasks – large datasets, bootstrap algorithm.

Applications to systemic and cross-market risks are explored in the article [18], where dependence between energy markets is analyzed, illustrating the importance of capturing joint tail behavior, solidifying the motivation for combination of copulas. This is extended in the work [19], which studies extreme dependence and spillover effects in green finance, emphasizing tail-driven risk transmission.

High-dimensional dependence modeling is addressed in the work [20] through pair-copula constructions (vine copulas), offering a scalable alternative to classical multivariate copulas which in turn can be used in patchwork setting yet with severe limitation of increased dimensionality and more complicated control of boundary condition satisfaction. This framework is operationalized in the work [21], which provides practical methodologies for fitting and analyzing vine copula models in complex datasets.

Domain-specific applications are demonstrated in article [22], where copula-based CVaR models are used for agricultural risk diversification, showing the adaptability of copulas in portfolio optimization problems. Finally, [23] introduces Gaussian mixture copulas as a modern approach to simultaneously model dependence in both central and tail regions, aligning closely with the objectives of patchwork constructions in achieving local flexibility. The framework used there is fundamental and creates a pair of two approaches together with work [4] – mixture models and patchworking when the composite copula generation problem arises.

3. Materials and Methods

3.1. Copula Definition

The main mathematical tool of this publication is a copula model. A copula is a function that blends marginal distributions in order to obtain the joint distribution between factors. The number of factors which can be fitted via copula models in principle is not limited, but the most common use case is bivariate.

The formal definition of a copula is as follows:

Let $[0,1]^2 \to [0,1]$.

The function C is a copula if:

$$C(u,0)=C(0,v)=0, \quad \forall u,v\in[0,1], \tag{1}$$

$$C(u,1)=u, \ C(1,v)=v, \quad \forall u,v\in[0,1], \tag{2}$$

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\geq 0, \tag{3}$$

for all $0\leq u_1\leq u_2\leq 1, \quad 0\leq v_1\leq v_2\leq 1$.

There are several distribution classes which can be used for modeling; the most common is the so-called Elliptical class – Gaussian copula and t-Student copulas for mostly symmetrical and weak tail dependency. For dependencies that display increasing correlation at the tails – at extremely small or extremely large values – Archimedean copulas are used: Frank, Gumbel and Clayton.

Gaussian:

Let $\rho\in(-1,1)$.

$$C_{\text{Gauss}}(u,v;\rho)=\Phi_\rho\left(\Phi^{-1}(u),\Phi^{-1}(v)\right), \tag{4}$$

where Φ is the standard normal CDF and Φ_ρ is the bivariate normal CDF with correlation ρ.

Clayton Copula:

Let $\theta \in (0, \infty)$.

$$C_{\mathrm{Cl}}(u, v; \theta) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{-1}{\theta}}, \tag{5}$$

where θ is the dependence parameter, which controls the strength of lower-tail dependence.

Gumbel Copula:

Let $\theta \in [1, \infty)$.

$$C_{\mathrm{Gu}}(u, v; \theta) = \exp\left(-\left((-\ln u)^{\theta} + (-\ln v)^{\theta}\right)^{\frac{1}{\theta}}\right), \tag{6}$$

where θ is the dependence parameter, which controls the strength of upper-tail dependence.

Frank Copula:

Let $\theta \in \mathbb{R} \setminus \{0\}$.

$$C_{\mathrm{Fr}}(u, v; \theta) = \frac{-1}{\theta} \ln\left(1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{e^{-\theta} - 1}\right), \tag{7}$$

where θ is the dependence parameter; the Frank copula is symmetric and exhibits no tail dependence.

3.2. Sklar's Theorem

Copula models take their formal origin and are largely based on Sklar's Theorem which postulates that under certain conditions, each continuous joint distribution is uniquely associated with a copula function.

Let $F : \mathbb{R}^2 \to [0,1]$ be a joint CDF with continuous marginals

$$F_X(x) = F(x, \infty), \quad F_Y(y) = F(\infty, y), \tag{8}$$

where F_X and F_Y are the marginal distribution functions of X and Y, then:

1. There exists a unique copula C such that:

$$F(x, y) = C\left(F_X(x), F_Y(y)\right), \tag{9}$$

where $x, y \in \mathbb{R}$ are the values of the modelled variables and the copula C is uniquely determined by the continuity of the marginals.

2. Conversely, for any copula C and marginals F_X, F_Y, the function above defines a valid joint CDF.

Continuity of F_X, F_Y implies uniqueness, which makes it possible to avoid practical problems during fitting procedure.

The result is profound but is not universal – in some cases there are several possible copula functions which stem from discontinuity of one or more marginal distributions, for example when one of the modelled factors is discrete in nature. Another issue is related to the true corresponding copula function. Its existence is known but, in most cases, the exact fitting copula is not known. In practical cases a substitute from a family which hosts copulas exhibiting similar properties is used, as can be derived from the data – in many cases based on the symmetry property. However, it is possible that no function can be located among known families and the copula itself needs to be approximated or specifically constructed.

3.3. Patchwork Copulas

One of the key results in copula theory is a widespread implementation of copula blending which makes it possible to combine several copulas on various parts of the distribution in one

mathematically sound model. This allows in many cases to overcome issues observed in models and use their strong sides to a great advantage. As was highlighted in literature overview, there is a close intersection between extreme value theory and modelling via copulas, but the results of the theory are valid only in asymptotic (extreme) regimes. But the main body of the distribution cannot be ignored. In cases like green projects, the extremes are of great importance but dependency in vicinity of median should not be overlooked too.

The issue prompts usage of several copulas which with regard to the validity of mathematical approaches in many cases is suitable only if the introduced copulas are combined in one. There are several approaches to how this can be achieved with different methods offering different advantages and disadvantages in terms of complexity, interpretation and limitations on the type of region splitting.

3.4. Mixture Copulas

Another approach to combining and building composite copula is a convex combination of several copulas which yields a single unified copula. The benefit of it compared to the patchwork approach is in simplicity of implementation – no region separation is required and weight which is assigned to each copula can be selected as an extra parameter.

The procedure is similar to building a simple copula – a function is selected and fitted to the data. This process is repeated for several copulas, and they are directly combined as weighted convex sum. There are properties of this convex combination which preserve some important features like continuity and density estimation. The overall handling of convergence and uniqueness is simpler than in case of patchwork since no control over marginal distributions or boundary matching is required.

In a formal way:

Let $C_B, C_T : [0,1]^2 \to [0,1]$ be bivariate copulas (bulk and tail components) and let $\alpha \in (0,1)$. define the mixture copula as follows:

$$\widetilde{C}(u,v) = \alpha C_T(u,v) + (1-\alpha) C_B(u,v), \tag{10}$$

where $u, v \in [0,1]$ are the marginal probabilities (pseudo-observations), C_B is the bulk copula, C_T is the tail copula and α is the mixing weight.

The obtained copula is a valid copula and can be used in the same manner as any standard copula.

If densities exist, then the mixture density is a convex combination of simple copula densities.

$$\tilde{c}(u,v) = \alpha c_T(u,v) + (1-\alpha) c_B(u,v), \tag{11}$$

where c_B and c_T are the densities of the bulk and tail copulas respectively.

3.5. Parameter Estimation and Model Selection

There are several approaches for estimating copulas parameters and thus building a copula model. Among the models it is then required to select the optimal one in some sense. Since the main emphasis is on probabilistic approach it is well regarded to select the model which yields the biggest likelihood of observed data. However, this approach cannot be used as it is since the mixture copula combines several models each of which is fitted to the data thus is more prone to overfitting. To mitigate issue of overfitting an appropriate modification is used which includes the number of parameters alongside the likelihood:

$$L(\theta) = \sum_{i=1}^{n} \log c(u_i, v_i; \theta), \tag{12}$$

where n is the number of observations, (u_i, v_i) is the i-th pair of pseudo-observations, c is the copula density and θ is the vector of copula parameters (for the mixture model: the weight α, the Gaussian correlation ρ and the tail copula parameter).

The parameter estimate is obtained by maximizing:

$$\hat{\theta} = \arg \max_{\theta \in \Theta} L(\theta), \tag{13}$$

where Θ is the admissible parameter space.

The model is selected based on:

$$\mathrm{AIC} = 2 \cdot \dim(\theta) - 2 L(\hat{\theta}), \tag{14}$$

where $\dim(\theta)$ is the number of estimated parameters; the model with the lowest AIC is selected.

3.6. Risk Estimation

The last step of the process is to calculate risk measures with the fitted copula. Risk is mostly associated with the tail behavior, and the computed enhanced estimates can be used for comparison for risk underestimation and overestimation. There are two metrics used in this work – Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). Those measures are well represented in the field and widely used, with the CVaR gaining a more prevalent position.

Formulas of measures are defined as follows.

VaR with confidence $\alpha \in (0,1)$:

$$\mathrm{VaR}_\alpha = \inf \{ l \in \mathbb{R} : F_L(l) \geq \alpha \}, \tag{15}$$

where L is the loss variable, F_L is its distribution function and α is the confidence level.

CVaR with confidence $\alpha \in (0,1)$:

$$\mathrm{CVaR}_\alpha = \mathrm{E}\left[L \middle| L > \mathrm{VaR}_\alpha \right], \tag{16}$$

where $\mathrm{E}[\cdot|\cdot]$ denotes the conditional expectation.

3.7. Dataset

A dataset used for this study has been already discussed in terms of copulas in our previous article [1] and used in a related study about green risks [2], which allows for a better and more concrete discussion of the improvement. In the preceding work, we utilized to a much greater extent a basic and most common approach in copula handling, focusing on selecting among common families of the Archimedean class and selecting suitable pairs which were selected with justification based on the AIC metric.

The distribution of features of interest is displayed in Table 1, which displays information regarding main statistics – mean, standard deviation, minimum/maximum, percentiles (25% – Q1, 50% – median, 75% – Q3).

Table 1
Distribution of key features of the dataset

Variable	Mean	Std	Min	Q1	Median	Q3	Max
Biodiversity	149,920	518,534	141.00	4893.50	19,428	76,027	7,562,284
Total Env. Intensity	-49.8788	343.97	-6247.62	-25.5600	-5.8600	-1.1475	397.07
Debt/Equity Ratio	2.0286	16.6871	-209.98	1.0690	1.9515	2.8937	195.03
Return on Equity	0.1600	1.3838	-15.8385	0.0540	0.1223	0.2181	22.8667

Yet in current study we continue pursuing the same dataset and using already obtained pairs. As was stated, mixture copulas might be treated as an enhanced version of basic copulas and build upon the same framework, in the case of two variables investigated, on a [0, 1] by [0, 1] support which is a normalized version of original support.

All variables are continuous so the utilization and statement of Sklar's theorem is quite straightforward and intuitive. Yet, there is an issue with missing data since it is quite common to miss some features per observation in such an intricate dataset as for green risks. Our approach was to pick only those rows which have all necessary information – no missing data. This approach can be examined in other works.

4. Experimental Results

The results can be separated into three parts related to the copula fitting procedure and analysis. The first step is to fit the single best copulas among Archimedean copulas which is shown in the table below.

The results of the fit are provided in Table 2.

Table 2
Best fit single copula

Pair	Best family	θ	AIC
Total Env. Intensity & Biodiversity	Frank	-6.5001	-409.09
Return on Equity & Debt/Equity Ratio	Gumbel	1.4427	-172.27
Return on Equity & Total Env. Intensity	Frank	1.4156	-30.07
Return on Equity & Biodiversity	Frank	-0.6537	-4.44
Debt/Equity Ratio & Biodiversity	Clayton	0.0772	-0.57
Debt/Equity Ratio & Total Env. Intensity	Frank	-0.1616	1.56

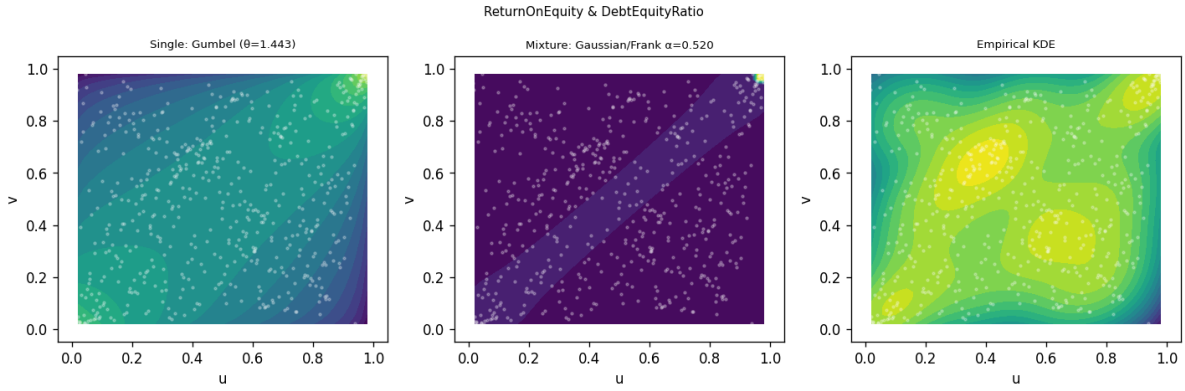
Three copula families were selected – Frank, Gumbel and Clayton. In 4 of 6 pairs, the best copula was the Frank copula. The variation of AIC is high, with some values remaining positive and some negative.

Table 3
Table of comparison of single copula and mixture model based on AIC

Pair	Single family	AIC (single)	Mixture family	AIC (mixture)	ΔAIC
Total Env. Intensity & Biodiversity	Frank	-409.09	Gaussian/Frank	-445.79	36.71
Return on Equity & Debt/Equity Ratio	Gumbel	-172.27	Gaussian/Frank	-24,208	24,036
Return on Equity & Total Env. Intensity	Frank	-30.07	Gaussian/Frank	-3355.92	3325.84
Return on Equity & Biodiversity	Frank	-4.44	Gaussian/Frank	-2450.92	2446.48
Debt/Equity Ratio & Biodiversity	Clayton	-0.57	Gaussian/Clayton	-6.12	5.55
Debt/Equity Ratio & Total Env. Intensity	Frank	1.56	Gaussian/Frank	-2445.19	2446.76

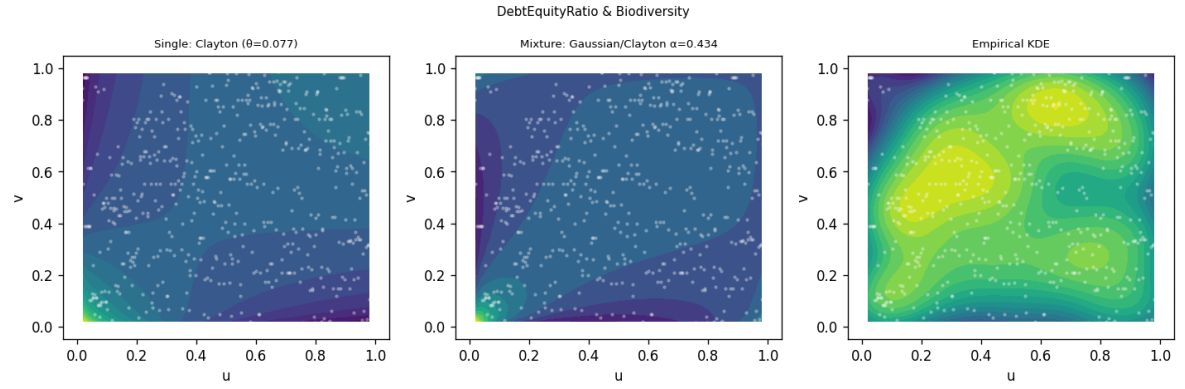
The next step after single copula fitting is to apply the mixture model fitting and to run comparison which is presented in Table 3.

The results have shown improvements across pairs when the mixture copula model was used with changes ranging from minor to significant – ΔAIC is 5.55 for Debt-to-Equity Ratio & Biodiversity and 24,036 for Return on Equity & Debt-to-Equity Ratio. To further illustrate change in density, the contour density plots are presented in Figures 1 and 2.



a) b) c)

Figure 1: Return on Equity & Debt-to-Equity Ratio: a) single copula fit; b) mixture copula; c) empirical KDE.



a) b) c)

Figure 2: Debt-to-Equity Ratio & Biodiversity: a) single copula fit; b) mixture copula; c) empirical KDE.

The introduction of new components for tail models changes the distribution. As can be seen in Figure 1, the introduction of an independent tail component resulted in a thinner distribution and less density beyond the main diagonal.

The opposite is true in Figure 2 where the density is allocated more evenly away from the main diagonal, which is closer to the local maxima displayed by the empirical plot.

The last step of the modeling experiment is to measure impact on risk metrics with introduction of composite modeling. Results are presented in Table 4.

The most notable change is for Debt-to-Equity Ratio paired with Biodiversity where the results differ by more than an order of magnitude. Such a great underestimation of risk is due to the initial fit change for both metrics, VaR and CVaR. Noteworthy is that the discrepancy grows non-linearly – roughly doubling at the 95% level and then jumping approximately 10-fold at the 99–99.5% level.

Table 4
Comparison of VaR and CVaR values for single and mixture model

Pair	Conf.	VaR (single)	CVaR (single)	VaR (mixture)	CVaR (mixture)
Total Env. Intensity & Biodiversity	90.0%	0.1770	0.6910	0.1752	0.7026
	95.0%	0.3781	1.1208	0.3801	1.1470
	99.0%	1.2988	2.9371	1.3555	3.0413
	99.5%	1.8302	4.3806	2.1838	4.3879
Return on Equity & Debt/Equity Ratio	90.0%	-0.0001	0.1175	0.0263	0.1747
	95.0%	0.0074	0.2322	0.0489	0.3143
	99.0%	0.0608	1.0752	0.2104	1.2036
	99.5%	0.1467	2.0629	0.3858	2.1137
Return on Equity & Total Env. Intensity	90.0%	0.1947	0.7567	0.1738	0.6540
	95.0%	0.4069	1.2299	0.3756	1.0503
	99.0%	1.5012	3.2358	1.2494	2.6888
	99.5%	2.1274	4.7043	1.8826	3.8854
Return on Equity & Biodiversity	90.0%	0.0080	0.1193	0.0026	0.0884
	95.0%	0.0162	0.2274	0.0049	0.1732
	99.0%	0.0843	1.0063	0.0256	0.8258
	99.5%	0.1708	1.8908	0.0622	1.6130
Debt/Equity Ratio & Biodiversity	90.0%	0.0108	0.0229	0.0313	0.1977
	95.0%	0.0135	0.0340	0.0588	0.3528
	99.0%	0.0283	0.0993	0.2399	1.3292
	99.5%	0.0493	0.1623	0.4124	2.3335
Debt/Equity Ratio & Total Env. Intensity	90.0%	0.2199	0.7487	0.1580	0.6375
	95.0%	0.4401	1.1872	0.3377	1.0471
	99.0%	1.4522	2.9463	1.1123	2.9160
	99.5%	2.2283	4.0893	1.7872	4.4551

Another important result is that not only the underestimation but also the overestimation of risk is displayed in data – for Debt-to-Equity Ratio and Total Environmental Intensity, where the risk measure was overestimated and joint extremes are less common than initially suspected.

Changes in AIC are not linearly associated with changes in risk measures (VaR and CVaR). For example, although the Debt-to-Equity Ratio and Total Environmental Intensity pair shows one of the largest AIC improvements in Table 3, this does not lead to a proportionate change in risk metrics, which instead decrease.

5. Discussion of Results

Overall, AIC improvement when using mixture model is universal but highly variable in magnitude, ranging from 5.5 units (Debt-to-Equity Ratio and Biodiversity) to 24,036 units (Return on Equity and Debt-to-Equity Ratio). The gain is the largest where a single copula is forced into the greatest compromise between bulk and tail behavior, that is, when the weight is closest to 0.5.

The mixing weight α is structured by variable-type homogeneity – thus by the domain from which each parameter comes. Pairs of the same type – two financial ratios (Return on Equity with Debt-to-Equity Ratio: $\alpha = 0.52$) or two environmental metrics (Total Environmental Intensity with Biodiversity: $\alpha = 0.53$) – show high tail weight, indicating substantial tail co-movement. Cross-type

pairs show near-zero α (Return on Equity with Biodiversity: α = 0.033; Debt-to-Equity Ratio with Total Environmental Intensity: α = 0.003), reflecting predominantly Gaussian dependence.

A conclusion regarding which copula class is suitable for tail modeling can also be drawn from the results. Thus, the Frank copula is the optimal tail component in five of six pairs, consistent with symmetric tail dependence – joint extremes in either direction are equally probable. The exception is Debt-to-Equity Ratio paired with Biodiversity, where Clayton is selected, capturing lower-tail concentration driven by the co-occurrence of leverage distress and ESG divestment.

Additionally, AIC improvement and risk metric change are not linearly related. Debt-to-Equity Ratio paired with Biodiversity records the smallest ΔAIC yet produces the largest tail risk shift: the 99.5th percentile CVaR under the mixture exceeds the single-copula estimate by a factor of 14. Conversely, Debt-to-Equity Ratio paired with Total Environmental Intensity shows a large ΔAIC of 2,447 but a change in risk metrics of less than ten percent. The decoupling arises because a large likelihood gain in the bulk region does not necessarily imply that the single-copula tail was misspecified.

Finally, the direction of the single-copula error is also pair-dependent. For Debt-to-Equity Ratio and Biodiversity, the single copula underestimates joint tail risk; for Debt-to-Equity Ratio and Total Environmental Intensity, it overestimates. No uniform correction factor can account for both cases, making pair-specific mixture estimation necessary for reliable tail risk measurement.

6. Conclusions

This article introduced and applied a mixture copula model to the analysis of green financial risk – the joint dependence between financial performance indicators and environmental metrics at the firm level. The model is defined as a fixed-weight convex combination of a Gaussian bulk copula and a parametric tail copula.

The model was applied to six pairs of variables – Return on Equity, Debt-to-Equity Ratio, Total Environmental Intensity, and Biodiversity exposure – drawn from a cross-sectional sample of 846 company-level observations (570 after listwise deletion of missing values). For each pair, the three parameters (α, the Gaussian correlation ρ, and the tail copula parameter θ) were estimated jointly by maximum likelihood, with Frank, Gumbel, and Clayton as competing tail families. Model selection was based on the Akaike Information Criterion.

The results are promising and can be further used in the context of modelling extreme events as a sign that combined models outperform simple models even accounting for extra parameters introduced. The comparison to the previous study highlighted the importance of an augmented copula structure, which better handles fitting of main observations (bulk) and the tail.

The study can be further extended to data with time varying components – how stable our results are over time and how the estimation depends on some specific observations in the data – how stable it is under bootstrap runs on the same sample.

Declaration on Generative AI

During the preparation of this work, the authors used GPT-5.4 and Grammarly in order to: Text translation, Grammar and spelling check. After using these tools, the authors reviewed and edited the content as needed and took full responsibility for the publication's content.

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