An Only Knowing Approach to Defeasible Description Logics (extended abstract)

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1 Introduction

We propose the only-knowing logic $\mathbf{O}^R\mathcal{ALC}$, based on the description logic $\mathcal{ALC}$. Only-knowing logics were originally designed as modal logics over a classical consequence relation for the representation of autoepistemic reasoning [17, 18] and default logic [15]. Only-knowing logics are worth considering for several reasons: They are monotonic logics with a clear separation between object level and meta level concepts, and they allow faithful and modular encodings of autoepistemic and default theories which do not increase the size of the representations. The encodings thus provide autoepistemic and default logics with formal semantics and conceptual clarity. Besides being completely axiomatized, *propositional* only-knowing logics support encodings of propositional autoepistemic and default logics with nice computational properties.

- There is a simple rewrite procedure that determines extensions. Some of the rewrite rules are preconditioned by SAT tests, and these are the only reference to meta-logical concepts in the procedure.
- The extension problem for propositional only-knowing logics have the same complexity as for propositional autoepistemic and default logics.

To our knowledge, no extension of only-knowing logics to description logics have yet been given. In this paper we take the description logic $\mathcal{ALC}$ as the underlying logic instead of classical propositional logic and generalize the machinery originally designed for propositional logics of only-knowing. The strength of the proposed only-knowing logic is the simple rewrite procedure that it admits for computing extensions, and the fact that reasoning is not harder than in $\mathcal{ALC}$. The logic $\mathbf{O}^R\mathcal{ALC}$ proposed in this paper subsumes the propositional logic that we proposed in [20] and extends that work in a non-trivial way.

The logic $\mathcal{ALCK}_{\mathcal{NF}}$ introduced by Donini, Nardi and Rosati [9] is closely related to $\mathbf{O}^R\mathcal{ALC}$. It is constructed by combining MKNF with $\mathcal{ALC}$. It is not obvious how this should be done, and in our adaptation of only-knowing logic to $\mathcal{ALC}$, we have been guided by $\mathcal{ALCK}_{\mathcal{NF}}$. Unlike [9], we do not treat so-called subjectively quantified expressions in this paper but the logic we present is strong enough to represent, e.g., default theories for $\mathcal{ALC}$.

In [9], a tableau procedure for $\mathcal{ALCK}_{\mathcal{NF}}$ is introduced. Although the procedure has been simplified [13], the procedure does not seem to be well suited for computation of extensions of default theories. In this case one will, it seems, have to guess extensions and then use the proof procedure to check whether or not the guess was correct. In contrast, the procedure that we propose determines all extensions directly. Unlike the proof system for $\mathcal{ALCK}_{\mathcal{NF}}$, we formalize inference steps that are sufficient for determining extensions but not for characterizing the whole semantics.

2 $\mathbf{O}^R\mathcal{ALC}$

The language of $\mathbf{O}^R\mathcal{ALC}$ is defined in two steps: first a *concept* language, then a modal *formula* language.

Concept Language. The basis for the concept language is $\mathcal{ALC}$ [1, 29]:

$$C, D \longrightarrow \top \mid \bot \mid C_{at} \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists R_{at}.C \mid \forall R_{at}.C$$

where C_{at} is an atomic concept and R_{at} an atomic role. We will call $\mathcal{ALC}$ concepts *objective* concepts. The concept language for $\mathbf{O}^R\mathcal{ALC}$ extends objective concepts with modal concept formation operators[1] $\mathbf{B}$ (belief) and $\mathbf{A}$ (assumption):

$$C, D \longrightarrow C_{\mathcal{ALC}} \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \mathbf{B}C \mid \mathbf{A}C$$

where $C_{\mathcal{ALC}}$ is an objective concept. A *modal concept* is of the form $\mathbf{B}C$ or $\mathbf{A}C$; if C is objective, the modal concept is *prime*. A concept is *subjective* if every objective subconcept is within the scope of a modal concept formation operator.

An *interpretation* $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ *over* Δ consists of a non-empty *domain* Δ and an *interpretation function* $\cdot^{\mathcal{I}}$ mapping atomic concepts and roles to subsets of Δ and $\Delta \times \Delta$ resp. Following [9] we assume that the domain contains all individuals in the language, and that $a^{\mathcal{I}} = a$ for each individual a. Concept descriptions are interpreted relative to a pair $\mathcal{M} = (U, V)$ and an interpretation $\mathcal{I}$; we require that U and V are interpretations over Δ and that $V \subseteq U$, but we do not require that $\mathcal{I} \in U$. $\mathcal{ALC}$ concepts evaluate relative to an $\mathcal{I}$ as usual, e.g., $(\neg C)^{\mathcal{M},\mathcal{I}} = \Delta \setminus C^{\mathcal{M},\mathcal{I}}$, while modal concepts are interpreted as follows:

$$(\mathbf{B}C)^{\mathcal{M},\mathcal{I}} = \bigcap\nolimits_{\mathcal{J} \in U} C^{\mathcal{M},\mathcal{J}} \qquad (\mathbf{A}C)^{\mathcal{M},\mathcal{I}} = \bigcap\nolimits_{\mathcal{J} \in V} C^{\mathcal{M},\mathcal{J}}$$

An *ABox* $\mathcal{A}$ is a finite set of *objective* membership assertions, i.e. *concept assertions* $C(a)$ and *role assertions* $R(a, b)$ for individuals a and b. $\mathcal{O}_{\mathcal{A}}$ is the set of individuals explicitly mentioned in $\mathcal{A}$. A *modal atom* is an assertion of the form $M(a)$, where M is a modal concept; $M(a)$ is *prime* if M is prime. $\mathcal{M}$ *satisfies* $C(a)$ *in* $\mathcal{I}$ if $a \in C^{\mathcal{M},\mathcal{I}}$; $\mathcal{M}$ satisfies $R(a, b)$ in $\mathcal{I}$ if $(a, b) \in R^{\mathcal{M},\mathcal{I}}$. As we will see, modal atoms play an essential role in the rewrite system. A *DBox* is a finite set of *terminological axioms*, i.e. *inclusions* $C \sqsubseteq D$ for concepts C and D, while

[1] In the literature, $\mathbf{K}$ is sometimes used instead of $\mathbf{B}$, and $\neg\mathbf{M}\neg$ instead of $\mathbf{A}$. For intuition about the modalities, consult the literature about MKNF [9, 22, 23] and only-knowing [15, 20].

a *TBox* is an objective DBox. $\mathcal{M}$ satisfies $C \sqsubseteq D$ in $\mathcal{I}$ if $C^{\mathcal{M},\mathcal{I}} \subseteq D^{\mathcal{M},\mathcal{I}}$. We define a *knowledge base (KB)* as a tuple $(\mathcal{T}, \mathcal{A}, \mathcal{D})$, where $\mathcal{T}$ is a TBox, $\mathcal{A}$ an ABox, and $\mathcal{D}$ is a DBox. The DBox will typically (but not necessarily) contain representation of default rules. We assume standard notions and notations for default theories. Let α, β_k for $1 \leqslant k \leqslant n$ and γ be $\mathcal{ALC}$-concepts. Below is an *$\mathcal{ALC}$-default* and its representation (following Konolige [14]; note that [9] has a $\mathbf{B}$ in front of γ) as an inclusion statement:

$$\alpha : \beta_1, \ldots, \beta_n \,/\, \gamma \quad \rightsquigarrow \quad \mathbf{B}\alpha \sqcap \neg\mathbf{A}\neg\beta_1 \sqcap \cdots \sqcap \neg\mathbf{A}\neg\beta_n \sqsubseteq \gamma$$

$\mathbf{B}\alpha$ denotes that α is *believed*, while $\neg\mathbf{A}\neg\beta$ denotes that β is considered *possible*, or that β is *consistent* with ones beliefs. $\mathbf{B}$ is stronger than $\mathbf{A}$ in the sense that every $\mathcal{M}$ satisfies $\mathbf{B}C \sqsubseteq \mathbf{A}C$ in every interpretation $\mathcal{I}$.

Interpretation of an objective concept is independent of $\mathcal{M}$, hence we may write $C^{\mathcal{I}}$ for $C^{\mathcal{M},\mathcal{I}}$ and say that $\mathcal{I}$ *satisfies* $C(a)$, written $\mathcal{I} \Vdash C(a)$, if $a \in C^{\mathcal{I}}$; similarly for roles, role assertions and inclusion statements. Hence we may write $\mathcal{I} \Vdash \mathcal{A}$ and $\mathcal{I} \Vdash \mathcal{T}$ if every element in the respective ABox and TBox is satisfied by $\mathcal{I}$. We also write $\mathcal{I} \Vdash (\mathcal{T}, \mathcal{A})$ if $\mathcal{I} \Vdash \mathcal{T}$ and $\mathcal{I} \Vdash \mathcal{A}$. For an objective assertion ϕ, we write $\mathcal{T}, \mathcal{A} \Vdash \phi$ if $\mathcal{I} \Vdash \phi$ for every $\mathcal{I}$ such that $\mathcal{I} \Vdash (\mathcal{T}, \mathcal{A})$. Similarly, $\mathcal{T}, \mathcal{A} \Vdash \mathcal{A}'$ if $\mathcal{I} \Vdash \mathcal{A}'$ for every $\mathcal{I}$ such that $\mathcal{I} \Vdash (\mathcal{T}, \mathcal{A})$.

Formula Language. Formulae are defined as follows. Concept and role assertions, T and F are *atomic* formulae. $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ are formulae if φ and ψ are. $\mathbf{B}$, $\mathbf{A}$, $\check{\mathbf{B}}$ and $\mathbf{O}$ are *modal operators*. $\mathbf{B}$ and $\mathbf{A}$ correspond to their concept formation operator counterparts, $\mathbf{O}$ is an "only knowing"-operator [17, 33, 20], while $\check{\mathbf{B}}$ is a "knowing at most"-operator, corresponding to $\mathbf{C}\neg$ of [20]. $L\varphi$ is a formula if L is a modal operator and φ is a formula without any occurrence of $\boxdot$. $\boxdot\varphi$ is a formula if φ is a *subjective* formula, i.e. every atomic formula that occurs as a subformula in φ is either a subjective assertion or within the scope of a modal operator. Note in particular that if C is an objective concept, $\mathbf{B}C(a)$ is a modal atom, and hence an atomic formula, while $\mathbf{B}(C(a))$ is not an atomic formula. Abbreviations: $\varphi \Rightarrow \psi$ and $\varphi \Leftrightarrow \psi$ are defined as usual; $\diamond\varphi$ is $\neg\boxdot\neg\varphi$; $\mathbf{O}^R\varphi$ is $\mathbf{O}\varphi \wedge \boxdot\neg\mathbf{O}\varphi$. Let L be a modal operator. A formula is *L-free* if it does not contain L (neither as a modal operator nor as a concept formation operator) and *L-basic* if it is subjective and contains no other modality than L.

Relative to the universal set $\mathcal{U}$ of all interpretations over Δ that satisfy $\mathcal{T}$, a *model* is a pair (U, V) such that $V \subseteq U \subseteq \mathcal{U}$. The $\boxdot$ modality quantifies over models by means of a binary relation $>$. Let $\mathcal{M} = (U, V)$. $\mathcal{M}' > \mathcal{M}$ if $\mathcal{M}' = (U', U)$ for some $U' \supset U$; in which case we say that $\mathcal{M}'$ is *larger than* $\mathcal{M}$. Truth conditions are given relative to an interpretation $\mathcal{I}$, which needs not be in V. Atomic formulae and connectives are interpreted as one would expect, e.g., $\mathcal{M} \models_{\mathcal{I}} C(a)$ iff $a \in C^{\mathcal{M},\mathcal{I}}$, $\mathcal{M} \models_{\mathcal{I}} \mathsf{T}$ and $\mathcal{M} \not\models_{\mathcal{I}} \mathsf{F}$. The modal operators are interpreted as follows:

- $\mathcal{M} \models_{\mathcal{I}} \mathbf{B}\varphi$ iff $\mathcal{M} \models_{\mathcal{J}} \varphi$ for each $\mathcal{J} \in U$;
- $\mathcal{M} \models_{\mathcal{I}} \mathbf{A}\varphi$ iff $\mathcal{M} \models_{\mathcal{J}} \varphi$ for each $\mathcal{J} \in V$;
- $\mathcal{M} \models_{\mathcal{I}} \check{\mathbf{B}}\varphi$ iff $\mathcal{J} \in U$ for each $\mathcal{J} \in \mathcal{U}$ such that $\mathcal{M} \models_{\mathcal{J}} \varphi$;

- $\mathcal{M} \models_{\mathcal{I}} \mathbf{O}\varphi$ iff ($\mathcal{M} \models_{\mathcal{J}} \varphi$ if and only if $\mathcal{J} \in U$) for each $\mathcal{J} \in \mathcal{U}$;[2]
- $\mathcal{M} \models_{\mathcal{I}} \boxdot\varphi$ iff $\mathcal{M}' \models_{\mathcal{I}} \varphi$ for every $\mathcal{M}' > \mathcal{M}$.

We write $\mathcal{M} \models \varphi$ if $\mathcal{M} \models_{\mathcal{I}} \varphi$ for each $\mathcal{I} \in \mathcal{U}$. Relative to a model $\mathcal{M}$, $\|\varphi\|^{\mathcal{M}}$ denotes the *truth set* of φ in $\mathcal{M}$, i.e. $\{\mathcal{I} \in \mathcal{U} \mid \mathcal{M} \models_{\mathcal{I}} \varphi\}$. Note that if φ is objective, $\|\varphi\|^{\mathcal{M}}$ is given independently of $\mathcal{M}$, as it only depends on the points in $\mathcal{U}$. For any objective φ:

- $U = \|\varphi\|$ iff $(U, V) \models \mathbf{O}\varphi$;
- $V \subseteq \|\varphi\|$ iff $(U, V) \models \mathbf{A}\varphi$.
- $U \subseteq \|\varphi\|$ iff $(U, V) \models \mathbf{B}\varphi$;
- $U \supseteq \|\varphi\|$ iff $(U, V) \models \check{\mathbf{B}}\varphi$;

Also note that in the clauses that define truth for the modal operators, the interpretation $\mathcal{I}$ plays no active role in the definition. When φ is subjective, it is immediate that $\mathcal{M} \models_{\mathcal{I}} \varphi$ iff $\mathcal{M} \models \varphi$, i.e. we can safely skip the reference to $\mathcal{I}$. This is also the reason why the following observation holds.

Lemma 1. *For any subjective concept M,*

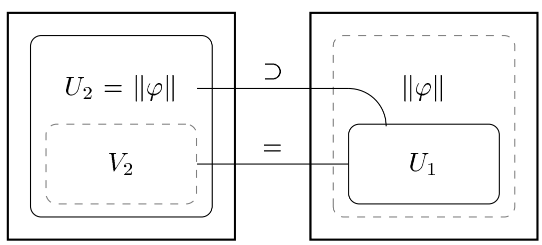
$$\text{either } \mathcal{M} \models M(a) \Leftrightarrow \top(a) \text{ or } \mathcal{M} \models M(a) \Leftrightarrow \bot(a).$$

It is also the case that $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$, for any subjective formula φ. A formula φ is *strongly valid*, written $\models\!\!\equiv \varphi$, if $\mathcal{M} \models \varphi$ for every model $\mathcal{M}$. There is also a weaker notion of validity, which is the notion of validity that we are primarily interested in. It is defined relative to the set of *weak models*: (U, V) is a weak model if $U = V$. φ is *valid*, written $\models \varphi$, if $\mathcal{M} \models \varphi$ for every weak model $\mathcal{M}$. Clearly, strong validity implies validity, but not conversely.

Lemma 2. $\models\!\!\equiv \mathbf{B}\varphi \Rightarrow \mathbf{A}\varphi$ *and* $\models \mathbf{A}\varphi \Rightarrow \mathbf{B}\varphi$.

Proof. Follows from the conditions $V \subseteq U$ (for arbitrary models) and $V = U$ (for weak models). □

We are interested in models $\mathcal{M}$ of $\mathbf{O}^R\varphi$, thus we want $\mathcal{M}$ to satisfy $\mathbf{O}\varphi$ but not $\diamond\mathbf{O}\varphi$, i.e. no larger model than $\mathcal{M}$ should also satisfy $\mathbf{O}\varphi$. The figure below illustrates the truth conditions relative to $\mathcal{M}_2 > \mathcal{M}_1$, where $\mathcal{M}_1 = (U_1, V_1)$ for an objective φ and an arbitrary $V_1 \subseteq U_1 \subset \|\varphi\|$, and $\mathcal{M}_2 = (U_2, V_2) = (\|\varphi\|, U_1)$.



$\mathcal{M}_2 \models \mathbf{O}\varphi \wedge \neg\diamond\mathbf{O}\varphi$ $\mathcal{M}_1 \models \neg\mathbf{O}\varphi \wedge \diamond\mathbf{O}\varphi$

Examining $\mathcal{M}_1$, we see that $U_1 \subset \|\varphi\|$, thus $\check{\mathbf{B}}\varphi$ does not hold in $\mathcal{M}_1$, hence neither does $\mathbf{O}\varphi$. $\mathbf{O}\varphi$ is, however, true in $\mathcal{M}_2$, and since $\mathcal{M}_2 > \mathcal{M}_1$, $\diamond\mathbf{O}\varphi$ is true in $\mathcal{M}_1$. Examining $\mathcal{M}_2$, we see that $U_2 = \|\varphi\|$, thus $\mathbf{O}\varphi$ is true. But as there is no $\mathcal{M} > \mathcal{M}_2$ that makes $\mathbf{O}\varphi$ true, $\diamond\mathbf{O}\varphi$ is not true. Hence $\mathbf{O}^R\varphi$ holds in $\mathcal{M}_2$.

[2] We could have defined $\mathbf{O}$ in terms of $\mathbf{B}$ and $\check{\mathbf{B}}$ (syntactically), as $\mathcal{M} \models_{\mathcal{I}} \mathbf{O}\varphi \Leftrightarrow (\mathbf{B}\varphi \wedge \check{\mathbf{B}}\varphi)$ but because of its special role in the rewrite system, this is not done.

The idea underlying the next lemma can be illustrated with the help of the model $\mathcal{M}_1$: Any model of $\diamondsuit\mathbf{O}\varphi$ must have the shape of $\mathcal{M}_1$, in which there must be a point $\mathcal{I} \notin U_1$ at which φ is true. Note that $\mathbf{B}\varphi$ and $\neg\check{\mathbf{B}}\varphi$ are both true in $\mathcal{M}_1$. Conversely, any model of $\mathbf{B}\varphi \land \neg\check{\mathbf{B}}\varphi$ must also have the shape of $\mathcal{M}_1$, satisfying $\diamondsuit\mathbf{O}\varphi$.

Lemma 3. *$\models\!\equiv \diamondsuit\mathbf{O}\varphi \Leftrightarrow (\mathbf{B}\varphi \land \neg\check{\mathbf{B}}\varphi)$ if φ is objective.*

For formulae in the $\mathbf{A}$-free fragment of the language, the two notions of validity coincide. It is easy to see that in this case the weak models of $\mathbf{O}^R\varphi$ are exactly the models (U, U) with the largest belief state U that satisfy $\mathbf{O}\varphi$.

Let $[\cdot]$ denote the function that replaces $\mathbf{A}$ with $\mathbf{B}$, and (for the service of the rewrite rules) puts the resulting formula on negation normal form, i.e. $[\mathbf{A}\varphi] = \mathbf{B}[\varphi]$, $[\neg\mathbf{A}\varphi] = \neg\mathbf{B}[\varphi]$, $[\neg\neg\varphi] = [\varphi]$, $[\neg(\varphi \land \psi)] = [\neg\varphi] \lor [\neg\psi]$, and so forth.

Lemma 4. *$\models\!\equiv \diamondsuit\varphi \Rightarrow [\varphi]$ if φ is $\mathbf{A}$-basic.*

Proof. Let $\mathcal{M} = (U, V)$. If $\mathcal{M} \models \diamondsuit\varphi$, then $\mathcal{M}' \models \varphi$ for some $\mathcal{M}' > \mathcal{M}$. By definition, $\mathcal{M}' = (W, U)$ for some $W \supset U$. Since φ is $\mathbf{A}$-basic, it is interpreted in $\mathcal{M}'$ only relative to U. But if we substitute $\mathbf{B}$ for all occurrences of $\mathbf{A}$ in φ, the resulting formula $[\varphi]$ is $\mathbf{B}$-basic and is hence interpreted in $\mathcal{M}$ relative to U in exactly the same way as φ is interpreted in $\mathcal{M}'$. Hence $\mathcal{M} \models [\varphi]$. □

We employ the convention that when a finite set of formulae X occurs in place of a formula, this is to be read as the conjunction of its elements, i.e. $\bigwedge X$; for $X = \emptyset$ this amounts to $\top$. Hence we will refer to a conjunction of objective assertions as an ABox. Next we show under which conditions $\mathbf{O}\mathcal{A}$, for some ABox $\mathcal{A}$, implies a prime modal atom or its negation. As we will see, this gives us the side conditions for the collapse rules in the rewrite system.

Lemma 5. *Let C and R be objective.*

1. *$\models\!\equiv \mathbf{O}\mathcal{A} \Rightarrow \mathbf{B}C(a)$ if $\mathcal{T}, \mathcal{A} \Vdash C(a)$;*
2. *$\models\!\equiv \mathbf{O}\mathcal{A} \Rightarrow \neg(\mathbf{B}C(a))$ if $\mathcal{T}, \mathcal{A} \nVdash C(a)$.*

Proof. Let $\mathcal{M} = (U, V)$ be a model such that $\mathcal{M} \models \mathbf{O}\mathcal{A}$. Then $U = \|\mathcal{A}\|$. *1.* Assume that $\mathcal{T}, \mathcal{A} \Vdash C(a)$. Then $\mathcal{I} \Vdash C(a)$ for every $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$. Hence $U \subseteq \|C(a)\|$. It follows that $\mathcal{M} \models \mathbf{B}C(a)$. *2.* Assume that $\mathcal{T}, \mathcal{A} \nVdash C(a)$. Then $\mathcal{I} \Vdash \neg C(a)$ for some $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$, hence there is some interpretation $\mathcal{J} \in U \cap \|\neg C(a)\|$. Hence $\mathcal{M} \models \neg(\mathbf{B}C(a))$. □

Corollary 1. *Let C and R be objective.*

1. *$\models\!\equiv \mathbf{O}\mathcal{A} \Rightarrow \mathbf{A}C(a)$ if $\mathcal{T}, \mathcal{A} \Vdash C(a)$;*
2. *$\models \mathbf{O}\mathcal{A} \Rightarrow \neg(\mathbf{A}C(a))$ if $\mathcal{T}, \mathcal{A} \nVdash C(a)$*

Proof. By Lemmata 2 and 5. □

Lemma 6. *Let $\mathcal{A}$ and $\mathcal{A}'$ be ABoxes.*

1. *$\models\!\equiv \mathbf{O}\mathcal{A} \Rightarrow \check{\mathbf{B}}\mathcal{A}'$ if $\mathcal{T}, \mathcal{A}' \Vdash \mathcal{A}$.*
2. *$\models\!\equiv \mathbf{O}\mathcal{A} \Rightarrow \neg\mathbf{B}\mathcal{A}'$ if $\mathcal{T}, \mathcal{A} \nVdash \mathcal{A}'$.*

3 The Rewrite System

Generalizing the rewrite system for the underlying propositional language in [20], the system in Fig. 1 consists of two rewrite relations on formulae. The rules of the $\rightarrow$ relation are based on strong equivalences, whereas the $\hat{\rightarrow}$ relation extends $\rightarrow$ with rules whose underlying equivalences are merely weak. We say that φ *reduces to* ψ if $\varphi \twoheadrightarrow \psi$, where $\twoheadrightarrow$ is the reflexive transitive closure of $\rightarrow$. Reduction can be performed on any subformula. A formula φ is on *normal form wrt.* $\rightarrow$ if there is no formula ψ such that $\varphi \rightarrow \psi$. The same notation is used for the $\hat{\rightarrow}$ relation. Reduction is performed modulo commutativity and associativity of $\wedge$ and $\vee$, and φ is identified with $\varphi \wedge \mathsf{T}$ and $\varphi \vee \mathsf{F}$; this implies that T and F behave as empty conjunction and disjunction resp. We define $\rightarrow$ to be the set of rules $l \rightarrow r$ in Fig. 1, while $\hat{\rightarrow}$ is the *union* of $\rightarrow$ and the set of rules $l \hat{\rightarrow} r$. Applying the $\rightarrow$ relation exhaustively before the $\hat{\rightarrow}$ is applied guarantees a correct rewrite process.

For formulae ν and μ, $\langle \nu/\mu \rangle$ is a *substitution function*: $\varphi\langle \nu/\mu \rangle$ denotes the result of substituting every occurrence of ν in φ with μ. Substitution is performed strictly on the formula level for the reason that assertions do not consist of subassertions in the sense that formulae consist of subformulae. A substitution of a value for an assertion $C(a)$ will not apply to, e.g., the assertion $C \sqcup D(a)$ but will apply to the equivalent formula $C(a) \vee D(a)$. The substitution function $\langle M(a)/V(a) \rangle$ for a prime modal atom $M(a)$ and $V \in \{\top, \bot\}$ is a *binding*, which *binds* $M(a)$ to $V(a)$.

The expand rule works by binding prime modal atoms in $\mathbf{O}\varphi$, assuming no subformula of φ is of the form $\mathbf{B}\psi$ or $\mathbf{A}\psi$ for an objective *formula* ψ. A formula with this property is objective if no prime modal atoms occur in it. Observe that the prime modal atom $\mathbf{B}C(a)$ has this property, whereas $\mathbf{B}(C(a))$ do not. Bindings might break this property, e.g., $\mathbf{B}(\mathbf{B}C(a))\langle \mathbf{B}C(a)/\top(a) \rangle = \mathbf{B}(\top(a))$. For this reason we have the M_4 rules which regain the property, should it be lost in the course of a binding operation, i.e. after the expand rule has been applied: $\mathbf{B}(\top(a)) \rightarrow (\mathbf{B}\top)(a)$. When there are no prime modal atoms left in $\mathbf{O}\varphi$, one may apply the collapse rules to reduce (or *collapse*) $\mathbf{O}\varphi \wedge M(a)$ to either $\mathbf{O}\varphi$ or F.

The last two rules of C_1 are strictly in $\hat{\rightarrow}$. These do not preserve strong equivalence and are hence not sound in all contexts. Applying the $\rightarrow$ relation exhaustively to a formula $\mathbf{O}^R\varphi$ results in formulae of a form which reflects that the last two rules of C_1 have not been applied. To characterize this we say that a formula is *semi-normal* if it is of the form $\mathbf{O}\mathcal{A} \wedge \Phi$ for an ABox $\mathcal{A}$ and a possibly empty set Φ of formulae of the form $\mathbf{A}C(a)$ and $\neg(\mathbf{A}C(a))$ with C objective and $\mathcal{T}, \mathcal{A} \not\Vdash \neg C(a)$.

Lemma 7. *For each semi-normal* $\mathbf{O}\mathcal{A} \wedge \Phi$*,*

1. $\mathbf{O}\mathcal{A} \wedge \Phi$ *is on normal form wrt.* $\rightarrow$*;*
2. *either* $\mathbf{O}\mathcal{A} \wedge \Phi \hat{\twoheadrightarrow} \mathbf{O}\mathcal{A}$ *or* $\mathbf{O}\mathcal{A} \wedge \Phi \hat{\twoheadrightarrow} \mathsf{F}$.

The primary function of the system is to reduce formulae of the form $\mathbf{O}\varphi$ and $\mathbf{O}^R\varphi$ for a conjunction of assertions φ, into a disjunction where each disjunct is

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Rules for reducing $\mathbf{O}$

The expand rule:

$$(M_1) \qquad \mathbf{O}\varphi \to (\mathbf{O}\varphi\langle M(a)/\top(a)\rangle \wedge M(a)) \vee (\mathbf{O}\varphi\langle M(a)/\bot(a)\rangle \wedge \neg(M(a)))$$
for any prime modal atom $M(a)$ occurring in φ

The domination and distribution rules:

$$(M_2) \qquad \varphi \wedge \mathsf{F} \to \mathsf{F}$$
$$(M_3) \qquad (\varphi \vee \mu) \wedge \psi \to (\varphi \wedge \psi) \vee (\mu \wedge \psi)$$

The assertional rules:

$$(M_4) \qquad \begin{array}{c} \neg(C(a)) \to (\neg C)(a) \\ \mathbf{B}(C(a)) \to (\mathbf{B}C)(a) \\ \mathbf{A}(C(a)) \to (\mathbf{A}C)(a) \end{array} \qquad \begin{array}{c} C(a) \wedge D(a) \to (C \sqcap D)(a) \\ C(a) \vee D(a) \to (C \sqcup D)(a) \end{array}$$

For $\phi_B = \mathbf{B}C(a)$ and $\phi_A = \mathbf{A}C(a)$:

$$(C_1) \qquad \begin{array}{rlrl} \mathbf{O}\mathcal{A} \wedge \phi_B \to \mathbf{O}\mathcal{A} & \text{if } \mathcal{T}, \mathcal{A} \Vdash C(a) & \mathbf{O}\mathcal{A} \wedge \phi_B \to \mathsf{F} & \text{if } \mathcal{T}, \mathcal{A} \nVdash C(a) \\ \mathbf{O}\mathcal{A} \wedge \neg\phi_B \to \mathsf{F} & \text{if } \mathcal{T}, \mathcal{A} \Vdash C(a) & \mathbf{O}\mathcal{A} \wedge \neg\phi_B \to \mathbf{O}\mathcal{A} & \text{if } \mathcal{T}, \mathcal{A} \nVdash C(a) \\ \mathbf{O}\mathcal{A} \wedge \phi_A \to \mathbf{O}\mathcal{A} & \text{if } \mathcal{T}, \mathcal{A} \Vdash C(a) & \mathbf{O}\mathcal{A} \wedge \phi_A \mathrel{\hat{\to}} \mathsf{F} & \text{if } \mathcal{T}, \mathcal{A} \nVdash C(a) \\ \mathbf{O}\mathcal{A} \wedge \neg\phi_A \to \mathsf{F} & \text{if } \mathcal{T}, \mathcal{A} \Vdash C(a) & \mathbf{O}\mathcal{A} \wedge \neg\phi_A \mathrel{\hat{\to}} \mathbf{O}\mathcal{A} & \text{if } \mathcal{T}, \mathcal{A} \nVdash C(a) \end{array}$$

For an ABox $\mathcal{A}'$:

$$(C_2) \qquad \begin{array}{rlrl} \mathbf{O}\mathcal{A} \wedge \check{\mathbf{B}}\mathcal{A}' \to \mathbf{O}\mathcal{A} & \text{if } \mathcal{T}, \mathcal{A}' \Vdash \mathcal{A} & \mathbf{O}\mathcal{A} \wedge \check{\mathbf{B}}\mathcal{A}' \to \mathsf{F} & \text{if } \mathcal{T}, \mathcal{A}' \nVdash \mathcal{A} \\ \mathbf{O}\mathcal{A} \wedge \neg\mathbf{B}\mathcal{A}' \to \mathsf{F} & \text{if } \mathcal{T}, \mathcal{A} \Vdash \mathcal{A}' & \mathbf{O}\mathcal{A} \wedge \neg\mathbf{B}\mathcal{A}' \to \mathbf{O}\mathcal{A} & \text{if } \mathcal{T}, \mathcal{A} \nVdash \mathcal{A}' \end{array}$$

Rules for reducing $\mathbf{O}^R$ *and* $\boxdot$

$$(R_1) \qquad \mathbf{O}^R\varphi \to \mathbf{O}\varphi \wedge \boxdot\neg\mathbf{O}\varphi$$
$$(R_2) \qquad \boxdot\neg(\varphi \vee \psi) \to \boxdot\neg\varphi \wedge \boxdot\neg\psi$$
$$(R_3) \qquad \boxdot\neg\mathsf{F} \to \mathsf{T}$$
$$(R_4) \qquad \boxdot\neg(\mathbf{O}\mathcal{A} \wedge \Phi) \to \neg\mathbf{B}\mathcal{A} \vee \check{\mathbf{B}}\mathcal{A} \vee [\neg\Phi] \text{ for semi-normal } \mathbf{O}\mathcal{A} \wedge \Phi$$

If Φ in rule R_4 is empty, we get

$$(R_4') \qquad \boxdot\neg\mathbf{O}\mathcal{A} \to \neg\mathbf{B}\mathcal{A} \vee \check{\mathbf{B}}\mathcal{A}$$

Fig. 1: ***The rules.*** *$\mathcal{A}$ is an ABox, C and D objective concepts, and R an objective role. The rules in C_k are called* collapse rules. *R_1 is not a proper rule, as the left hand side is an abbreviation of the right. We include it for readability reasons.*

of the form $\mathbf{O}\psi$ for some objective ψ. When φ is the formula representation of a knowledge base, each disjunct represents a Reiter extension of φ (when $\mathbf{O}^R$ is used) or an autoepistemic extension (when $\mathbf{O}$ is used). To achieve this, the rewrite system needs rules for reducing formulae prefixed with $\mathbf{O}$ and $\boxdot\neg$, and whatever they are reduced to.

Note that an objective formula may or may not be an assertion. $C(a) \vee \neg C(a)$ is, for instance, not an assertion, but can be transformed to an equivalent assertion if we "push the a outward" and change $\vee$ to $\sqcup$. Below we address the reverse operation of "pushing the a inward." We do this to get prime modal atoms as subformulae, so that the expand rule may be applied. This operation, $[\![\cdot]\!]$, is defined as follows. For role assertions, $[\![R(a,b)]\!] = R(a,b)$, and for concept assertions, $[\![C(a)]\!] = C(a)$ for $C \in \{\top, \bot\}$ or atomic, and

$$[\![LC(a)]\!] = \begin{cases} LC(a) & \text{if } C \text{ is objective} \\ \hat{L}[\![C(a)]\!] & \text{otherwise} \end{cases} \qquad \text{for } L \in \{\neg, \mathbf{B}, \mathbf{A}\};$$

$$[\![C \star D(a)]\!] = \begin{cases} C \star D(a) & \text{if } C \star D \text{ is objective} \\ [\![C(a)]\!] \mathbin{\hat{\star}} [\![D(a)]\!] & \text{otherwise} \end{cases} \qquad \text{for } \star \in \{\sqcap, \sqcup\},$$

where $\hat{\sqcap} = \wedge$ and $\hat{\sqcup} = \vee$, and $\hat{L} = L$ for $L \in \{\neg, \mathbf{B}, \mathbf{A}\}$. Inclusions are instantiated and translated into formulae as follows: For an individual a, $[\![C \sqsubseteq D]\!]_a = [\![C(a)]\!] \Rightarrow [\![D(a)]\!]$.

Example 1. Let us address the default $C : \top \,/\, C$, which is represented as $\mathbf{B}C \sqsubseteq C$. Let $\varphi = [\![\mathbf{B}C \sqsubseteq C]\!]_a = \mathbf{B}C(a) \Rightarrow C(a)$. To reduce $\mathbf{O}\varphi$, we first apply the expand rule and then rules from the C_1 group. The same reductions also apply in a boxed context. As the formula is $\mathbf{A}$-free, the $\hat{\rightarrow}$ relation will not be needed.

$$\begin{aligned} \mathbf{O}(\neg\mathbf{B}C(a) \vee C(a)) &\rightarrow (\mathbf{O}C(a) \wedge \mathbf{B}C(a)) \vee (\mathbf{O}\top(a) \wedge \neg\mathbf{B}C(a)) && (M_1) \\ &\rightarrow \mathbf{O}C(a) \vee \mathbf{O}\top(a) && (C_1) \\ \boxdot\neg\mathbf{O}(\neg\mathbf{B}C(a) \vee C(a)) &\twoheadrightarrow \boxdot\neg(\mathbf{O}C(a) \vee \mathbf{O}\top(a)) \\ &\rightarrow \boxdot\neg\mathbf{O}C(a) \wedge \boxdot\neg\mathbf{O}\top(a) && (R_2) \\ &\twoheadrightarrow (\neg\mathbf{B}C(a) \vee \check{\mathbf{B}}C(a)) \wedge (\neg\mathbf{B}\top(a) \vee \check{\mathbf{B}}\top(a)) && (R_4) \end{aligned}$$

Having reduced $\mathbf{O}\varphi$ and $\boxdot\neg\mathbf{O}\varphi$, we reduce $\mathbf{O}^R\varphi$.

$$\begin{aligned} \mathbf{O}^R\varphi &\rightarrow \mathbf{O}\varphi \wedge \boxdot\neg\mathbf{O}\varphi \\ &\twoheadrightarrow (\mathbf{O}C(a) \vee \mathbf{O}\top(a)) \wedge (\neg\mathbf{B}C(a) \vee \check{\mathbf{B}}C(a)) \wedge (\neg\mathbf{B}\top(a) \vee \check{\mathbf{B}}\top(a)) \\ &\rightarrow (\mathbf{O}C(a) \wedge (\neg\mathbf{B}C(a) \vee \check{\mathbf{B}}C(a)) \wedge (\neg\mathbf{B}\top(a) \vee \check{\mathbf{B}}\top(a))) \vee \\ &\quad\ (\mathbf{O}\top(a) \wedge (\neg\mathbf{B}C(a) \vee \check{\mathbf{B}}C(a)) \wedge (\neg\mathbf{B}\top(a) \vee \check{\mathbf{B}}\top(a))) && (M_3) \end{aligned}$$

Distributing conjunctions over disjunctions, using M_3, we obtain a formula on DNF, which reduces to $\mathbf{O}\top(a)$. This corresponds to the unique Reiter extension. □

4 The Modal Reduction Theorem

The Modal Reduction Theorem for $\mathbf{O}^R\mathcal{ALC}$ states that whenever a formula $\mathbf{O}^R\varphi$ encodes a knowledge base, it is logically equivalent to a disjunction, where each disjunct is of the form $\mathbf{O}\mathcal{A}$ for some ABox $\mathcal{A}$. In fact, each of these disjuncts has an essentially unique weak model. It is hence possible, within the logic itself, to decompose a formula $\mathbf{O}^R\varphi$ into a form which directly exhibits its models.

We translate an entire knowledge base $\Sigma = (\mathcal{T}, \mathcal{A}, \mathcal{D})$ into a formula as follows: $[\![\Sigma]\!]_J = \mathcal{A} \wedge [\![\mathcal{D}]\!]_J$, where $[\![\mathcal{D}]\!]_J = \{[\![C \sqsubseteq D]\!]_a \mid a \in J \;\&\; C \sqsubseteq D \in \mathcal{D}\}$ for some non-empty set of individuals $J \subseteq \Delta$. Observe that the TBox $\mathcal{T}$ seemingly disappears. It does, however, reappear in the side conditions of the collapse rules.

The Modal Reduction Theorem. *For each $\Sigma = (\mathcal{T}, \mathcal{A}, \mathcal{D})$, there are ABoxes $\mathcal{A}_1, \ldots, \mathcal{A}_n$ for some $n \geqslant 0$, such that $\mathcal{A} \subseteq \mathcal{A}_k$ for $1 \leqslant k \leqslant n$, and*

$$\models \mathbf{O}^R[\![\Sigma]\!]_{\mathcal{O}_\mathcal{A}} \Leftrightarrow (\mathbf{O}\mathcal{A}_1 \vee \cdots \vee \mathbf{O}\mathcal{A}_n).$$

Proof. By completeness (Theorem 1) and soundness (Theorems 2 and 3). □

We define the *extensions*[3] of Σ to be exactly the ABoxes $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in The Modal Reduction Theorem. Hence the notion of extension makes no appeal to the formula language, only the concept language.

The rewrite system is just strong enough for establishing the Modal Reduction Theorem: It is sound and complete for reductions of $\mathbf{O}^R$-formulae into disjunctions of the appropriate type. It is, however, not complete for the logic of $\mathbf{O}^R$ itself. From the point of view of computing default extensions this is enough, because only a subset of the logic of $\mathbf{O}^R$ is actually needed for the Modal Reduction Theorem.

Example 2. The $\mathcal{ALC}$-default $\mathsf{Employee} : \neg\mathsf{Manager} \,/\, \mathsf{Engineer} \sqcup \mathsf{Mathematician}$ (adapted from [9]) is represented as $\mathbf{B}\mathsf{Employee} \sqcap \neg\mathbf{A}\mathsf{Manager} \sqsubseteq (\mathsf{Engineer} \sqcup \mathsf{Mathematician})$. Let $\mathcal{D}$ consist of this inclusion, and let $J = \{Bob\}$:

$$\begin{aligned}
&\mathbf{O}^R(\mathcal{A} \wedge [\![\mathcal{D}]\!]_J)\\
=\;&\mathbf{O}^R(\mathcal{A} \wedge [\![\mathbf{B}\mathsf{Employee} \sqcap \neg\mathbf{A}\mathsf{Manager} \sqsubseteq \mathsf{Eng} \sqcup \mathsf{Mat}]\!]_{Bob})\\
=\;&\mathbf{O}^R(\mathcal{A} \wedge ([\![\mathbf{B}\mathsf{Employee} \sqcap \neg\mathbf{A}\mathsf{Manager}(Bob)]\!] \Rightarrow [\![\mathsf{Eng} \sqcup \mathsf{Mat}(Bob)]\!])\\
=\;&\mathbf{O}^R(\mathcal{A} \wedge ([\![\mathbf{B}\mathsf{Employee}(Bob)]\!] \wedge [\![\neg\mathbf{A}\mathsf{Manager}(Bob)]\!] \Rightarrow [\![\mathsf{Eng} \sqcup \mathsf{Mat}(Bob)]\!])\\
=\;&\mathbf{O}^R(\mathcal{A} \wedge (\mathbf{B}\mathsf{Employee}(Bob) \wedge \neg\mathbf{A}\mathsf{Manager}(Bob) \Rightarrow \mathsf{Eng} \sqcup \mathsf{Mat}(Bob)))
\end{aligned}$$

This formula can be reduced to a simpler form, depending on the ABox $\mathcal{A}$ and the TBox. Let $\mathcal{T} = \{\mathsf{Manager} \sqsubseteq \mathsf{Employee}\}$, and let $\phi_1 = \mathbf{B}\mathsf{Employee}(Bob)$, $\phi_2 = \mathbf{A}\mathsf{Manager}(Bob)$ and $\gamma = \mathsf{Engineer} \sqcup \mathsf{Mathematician}(Bob)$. For any $\mathcal{A}$ such that $\mathcal{A} \Vdash \mathsf{Employee}(Bob)$, $\mathbf{O}\mathcal{A} \wedge \phi_1 \to \mathbf{O}\mathcal{A}$ and $\mathbf{O}\mathcal{A} \wedge \neg\phi_1 \to \mathsf{F}$, hence

$$\mathbf{O}(\mathcal{A} \wedge (\phi_1 \wedge \neg\phi_2 \Rightarrow \gamma)) \twoheadrightarrow (\mathbf{O}\mathcal{A} \wedge \phi_1 \wedge \phi_2) \vee (\mathbf{O}(\mathcal{A} \wedge \gamma) \wedge \phi_1 \wedge \neg\phi_2) \vee$$

[3] Extensions are here taken in the sense of default logic, i.e. Reiter extensions; autoepistemic extensions (stable expansions) can be defined from the Modal Reduction Theorem for $\mathbf{O}$ in the same way.

$$(\mathbf{O}\mathcal{A} \wedge \neg\phi_1 \wedge \phi_2) \vee (\mathbf{O}\mathcal{A} \wedge \neg\phi_1 \wedge \neg\phi_2)$$
$$\twoheadrightarrow (\mathbf{O}\mathcal{A} \wedge \phi_2) \vee (\mathbf{O}(\mathcal{A} \wedge \gamma) \wedge \neg\phi_2).$$

For the ABox $\mathcal{A}_1 = \{\mathsf{Employee}(Bob)\}$, the reduct is on normal form wrt. $\to$ but not for the ABox $\mathcal{A}_2 = \{\mathsf{Manager}(Bob)\}$:

$$(\mathbf{O}\mathcal{A}_1 \wedge \phi_2) \vee (\mathbf{O}(\mathcal{A}_1 \wedge \gamma) \wedge \neg\phi_2) \hat{\twoheadrightarrow} \mathbf{O}(\mathcal{A}_1 \wedge \gamma)$$
$$(\mathbf{O}\mathcal{A}_2 \wedge \phi_2) \vee (\mathbf{O}(\mathcal{A}_2 \wedge \gamma) \wedge \neg\phi_2) \twoheadrightarrow \mathbf{O}\mathcal{A}_2$$

In the former case, Bob is an engineer or a mathematician, in the latter Bob is only a manager. Reducing $\mathbf{O}^R(\mathcal{A} \wedge [\![\mathcal{D}]\!]_J)$ produces the same extensions. □

4.1 Soundness and Completeness

Lemma 8. *For each $\Sigma = (\mathcal{T}, \mathcal{A}, \mathcal{D})$, there is a set $\Gamma_1 = \{\mathbf{O}(\mathcal{A} \wedge \mathcal{A}_k) \wedge \Phi_k\}_{1 \leqslant k \leqslant n}$ of semi-normal formulae for some $n \geqslant 0$ such that for some set $\Gamma_2 \subseteq \Gamma_1$*

$$(a)\ \ \mathbf{O}[\![\Sigma]\!]_J \twoheadrightarrow \bigvee \Gamma_1 \quad \textit{and} \quad (b)\ \ \mathbf{O}^R[\![\Sigma]\!]_J \twoheadrightarrow \bigvee \Gamma_2.$$

Theorem 1 (Completeness). *For each $\Sigma = (\mathcal{T}, \mathcal{A}, \mathcal{D})$, for some $n \geqslant 0$, there are ABoxes $\mathcal{A}_1, \dots, \mathcal{A}_n$ such that for some set Γ of semi-normal formulae,*

$$\mathbf{O}^R[\![\Sigma]\!]_{\mathcal{O}_\mathcal{A}} \twoheadrightarrow \bigvee \Gamma \hat{\twoheadrightarrow} (\mathbf{O}(\mathcal{A} \wedge \mathcal{A}_1) \vee \dots \vee \mathbf{O}(\mathcal{A} \wedge \mathcal{A}_n)).$$

Proof. By Lemmata 7 and 8. □

Lemma 9. *If $\mathcal{M} \models_\mathcal{I} (\nu \Leftrightarrow \mu)$ and ν does not occur within the scope of $\boxdot$ in φ, then $\mathcal{M} \models_\mathcal{I} (\varphi \Leftrightarrow \varphi\langle\nu/\mu\rangle)$.*

The previous result state the condition under which substitution of equivalents is valid. For substitution within the context of $\boxdot$ we need the stronger notion of validity. From the point of view of formula rewriting, the significance of strong validity is that it is required for general substitution of equivalents.

Lemma 10. $\models\!\!\!\equiv (\varphi \Leftrightarrow \varphi\langle\nu/\mu\rangle)$ *if* $\models\!\!\!\equiv (\nu \Leftrightarrow \mu)$.

Theorem 2 (Soundness of $\to$). *If $\varphi \twoheadrightarrow \psi$ then $\models\!\!\!\equiv \varphi \Leftrightarrow \psi$.*

Proof. By Lemma 10, it is sufficient to show that $\models\!\!\!\equiv l \Leftrightarrow r$ for each rule $l \to r$, in which case we say that the rule is strongly valid. Rule R_1 is trivial. R_2 follows from the fact that $\models\!\!\!\equiv \boxdot(\varphi \wedge \psi) \Leftrightarrow (\boxdot\varphi \wedge \boxdot\psi)$ and De Morgan's law, while R_3 follows from the fact that $\models\!\!\!\equiv \boxdot\mathsf{T}$. M_2 and M_3 are propositionally valid, M_4 is left to the reader. C_1 follows from Lemma 5 and Corollary 1, while C_2 follows from Lemma 6. The two remaining cases are treated in detail.

R_4 : We show that $\models\!\!\!\equiv \boxdot\neg(\mathbf{O}\varphi \wedge \Phi) \Leftrightarrow ((\mathbf{B}\varphi \wedge \neg\breve{\mathbf{B}}\varphi) \Rightarrow [\neg\Phi])$ for any semi-normal $\mathbf{O}\varphi \wedge \Phi$. By Lemma 3, $\models\!\!\!\equiv \Diamond\mathbf{O}\varphi \Leftrightarrow (\mathbf{B}\varphi \wedge \neg\breve{\mathbf{B}}\varphi)$, hence by Lemma 10, we have to show that $\models\!\!\!\equiv \boxdot\neg(\mathbf{O}\varphi \wedge \Phi) \Leftrightarrow (\Diamond\mathbf{O}\varphi \Rightarrow [\neg\Phi])$. ($\Rightarrow$) Assume that $\mathcal{M} \models \boxdot(\mathbf{O}\varphi \Rightarrow \neg\Phi)$ and $\mathcal{M} \models \Diamond\mathbf{O}\varphi$. Then $\mathcal{M} \models \Diamond(\mathbf{O}\varphi \wedge \neg\Phi)$, thus $\mathcal{M} \models \Diamond\neg\Phi$, hence $\mathcal{M} \models [\neg\Phi]$ by Lemma 4. ($\Leftarrow$) We show that $\mathcal{M} \models (\boxdot\neg\mathbf{O}\varphi \vee [\neg\Phi]) \Rightarrow \boxdot\neg(\mathbf{O}\varphi \wedge \Phi)$ for every $\mathcal{M}$. Now there are two cases. If $\mathcal{M} \models \boxdot\neg\mathbf{O}\varphi$, then $\mathcal{M} \models \boxdot\neg(\mathbf{O}\varphi \wedge \Phi)$. If $\mathcal{M} \models [\neg\Phi]$, then $\mathcal{M} \models \neg[\Phi]$, thus $\mathcal{M} \models \boxdot\neg\Phi$ by Lemma 4, thus $\mathcal{M} \models \boxdot\neg(\mathbf{O}\varphi \wedge \Phi)$.

M_1 : We show that $\models \mathbf{O}\varphi \Leftrightarrow (\mathbf{O}\varphi\langle M(a)/\top(a)\rangle \wedge M(a)) \vee (\mathbf{O}\varphi\langle M(a)/\bot(a)\rangle \wedge \neg(M(a)))$ for any prime modal atom $M(a)$. Let $\mathcal{M}$ be an arbitrary model. Then either $\mathcal{M} \models (M(a) \Leftrightarrow \top(a))$ or $\mathcal{M} \models (M(a) \Leftrightarrow \bot(a))$ by Lemma 1. By Lemma 9, as φ is $\boxdot$-free, either $\mathcal{M} \models \mathbf{O}\varphi \Leftrightarrow \mathbf{O}\varphi\langle M(a)/\top(a)\rangle$ or $\mathcal{M} \models \mathbf{O}\varphi \Leftrightarrow \mathbf{O}\varphi\langle M(a)/\bot(a)\rangle$, resp. In either case $\mathcal{M}$ satisfies either $(M(a) \Leftrightarrow \top(a)) \wedge (\mathbf{O}\varphi \Leftrightarrow \mathbf{O}\varphi\langle M(a)/\top(a)\rangle)$ or $(M(a) \Leftrightarrow \bot(a)) \wedge (\mathbf{O}\varphi \Leftrightarrow \mathbf{O}\varphi\langle M(a)/\bot(a)\rangle)$, from which the equivalence follows. □

Theorem 3 (Soundness of $\hat{\rightarrow}$). *For any disjunction φ of semi-normal formulae, if $\varphi \hat{\twoheadrightarrow} \psi$ then $\models \varphi \Leftrightarrow \psi$.*

Proof. As semi-normal formulae do not contain $\boxdot$, by Lemma 9, it is sufficient to show that $\models l \Leftrightarrow r$ for each rule $l \hat{\rightarrow} r$, in which case we say that the rule is valid. That the rules of $\rightarrow$ are valid is obvious, given that they are strongly valid. The remaining rules are in the C_1 group; validity follows from Corollary 1. □

4.2 Complexity

The *extension problem* is determining whether a KB has an extension. A redeeming feature of $\mathbf{O}^R\mathcal{ALC}$ is that the extension problem is not harder than the $\mathcal{ALC}$ problem of instance checking.

Theorem 4. *The extension problem of $\mathbf{O}^R\mathcal{ALC}$ is* PSPACE*-complete.*

Proof (Sketch). The translation from Σ to $[\![\Sigma]\!]_{\mathcal{O}_\mathcal{A}}$ can be done in polynomial time: $|\mathcal{O}_\mathcal{A}| \times |\mathcal{D}|$. The corresponding extension problem when the underlying logic is propositional is in $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$ as it can be solved nondeterministically with a polynomial number of calls to a propositional SAT oracle [33]. The main difference here is that instead of a SAT oracle, we need an oracle that can do instance checks in $\mathcal{ALC}$, which is PSPACE-complete [1]. Thus the extension problem is in $\mathrm{NP}^{\mathrm{PSPACE}}$, which is equal to PSPACE [25]. □

5 Future Work

The underlying concept language does not have to be $\mathcal{ALC}$. Other concept languages with lower complexity like $\mathcal{DL}$-Lite [4] might prove more useful in an actual implementation.

An natural extension is to introduce a partial order $(I, \preccurlyeq)$, intuitively representing confidence levels, and for each index $k \in I$, adding modal operators $\mathbf{B}_k$, $\mathbf{A}_k$, $\check{\mathbf{B}}_k$, $\mathbf{O}_k$, and $\boxdot_k$ to the signature of the logic, in order to represent ordered default theories. Another extension would be allowing subjectively quantified expressions, i.e. concepts of the form $\forall\mathbf{X}R.\mathbf{Y}C$ and $\exists\mathbf{X}R.\mathbf{Y}C$ for $\mathbf{X}, \mathbf{Y} \in \{\mathbf{B}, \mathbf{A}\}$, like [9] does.

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