Efficient Reasoning in Combinations of $\mathcal{EL}$ and (Fragments of) $\mathcal{FL}_0$

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Abstract. We study possibilities of combining (fragments) of the lightweight description logics $\mathcal{FL}_0$ and $\mathcal{EL}$, and identify classes of subsumption problems in a combination of $\mathcal{EL}$ and Horn-$\mathcal{FL}_0$, which can be checked in PSPACE resp. PTIME. Since $\mathcal{FL}_0$ allows universal role restrictions and $\mathcal{EL}$ allows existential role restrictions, we thus have a framework where subsumption between expressions including both types of role restrictions (but for disjoint sets of roles) can be checked in polynomial space or time.

1 Introduction

Description logics [5] are a family of knowledge representation formalisms that can model the terminological knowledge of a given domain; they are, for instance, the logical foundation of the W3C language for the Semantic Web. Their most interesting feature is that they aim at maximizing expressive power while retaining decidability. However, with the size of the ontologies appearing in many applications, decidability alone is not enough because the complexity of the reasoning procedures combined with the size of the ontologies makes reasoning too costly. This consideration triggered the development of lightweight sub-families of description logics. Among them, we mention $\mathcal{EL}$ (which only allows the use of conjunction and existential role restrictions) [1] and some of its extensions such as $\mathcal{EL}^+$ and $\mathcal{EL}^{++}$ [2, 4, 3]. These logics can model some very interesting domains sufficiently well to be used widely, for example in the SNOMED ontology [16]. Another lightweight description logic is $\mathcal{FL}_0$ (which only allows the use of conjunction and universal role restrictions). While subsumption without TBoxes in $\mathcal{FL}_0$ is decidable in PTIME, its subsumption problem is in PSPACE for standard terminologies and EXPTIME for general terminologies [8, 4]. Since some very interesting forms of knowledge require universal restrictions in order to be modeled adequately, recent research has identified tractable fragments of $\mathcal{FL}_0$, such as the Horn-$\mathcal{FL}_0$ fragment (defined by syntactic restrictions) for which the subsumption problem is in PTIME [9].

A combination of $\mathcal{EL}$ and (fragments of) $\mathcal{FL}_0$ is clearly interesting because of the added expressivity it offers. At the same time, if we allow an unrestricted combination we lose the lower complexity of the components. In this paper we

Table 1. Constructors and their semantics

Constructor name	Syntax	Semantics
negation	$\neg C$	$D^{\mathcal{I}} \backslash C^{\mathcal{I}}$
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction	$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
existential restriction	$\exists r.C$	$\{x \mid \exists y((x, y) \in r^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}})\}$
universal restriction	$\forall r.C$	$\{x \mid \forall y((x, y) \in r^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}})\}$

present a way to combine these description logics such that we can verify subsumption between two mixed concept expressions w.r.t. TBoxes efficiently, and identify situations in which this can be done in PSPACE, resp. PTIME.

Structure of the Paper. In Sect. 2 we give general definitions and introduce the description logics $\mathcal{ALC}, \mathcal{EL}$ and $\mathcal{FL}_0$ and their combination. Sect. 3 presents the algebraic semantics for each logic and their combination. Sect. 4 presents generalities on local theory extensions and hierarchical reasoning (which we use in our approach). These methods are used in Sect. 5, where we present possibilities of hierarchical reasoning in a combination of $\mathcal{EL}$ and (fragments of) $\mathcal{FL}_0$.

2 Description Logics

The central notions in description logics are concepts and roles. In any description logic a set N_C of *concept names* and a set N_R of *roles* is assumed to be given. Complex concepts are defined starting with the concept names in N_C, with the help of a set of *concept constructors*. The semantics of description logics is defined in terms of interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set, and the function $\cdot^{\mathcal{I}}$ maps each concept name $C \in N_C$ to a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and each role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Table 1 shows the constructor names used in $\mathcal{ALC}$ and their semantics. The extension of $\cdot^{\mathcal{I}}$ to concept descriptions is inductively defined using the semantics of the constructors.

Terminology. A *terminology* (TBox, for short) is a finite set of *primitive concept definitions* of the form $C \equiv D$, where C is a concept name and D a concept description; and *general concept inclusions* (GCI) of the form $C \sqsubseteq D$, where C and D are concept descriptions. A TBox which only contains primitive concept definitions and every concept name is defined at most once is called *standard*. (As definitions can be expressed as double inclusions, by TBox (or general TBox) we will refer to a TBox consisting of general concept inclusions only.) An interpretation $\mathcal{I}$ is a model of a TBox $\mathcal{T}$ if it satisfies:

- all concept definitions in $\mathcal{T}$, i.e. $C^{\mathcal{I}}{=}D^{\mathcal{I}}$ for all definitions $C{\equiv}D \in \mathcal{T}$;
- all general concept inclusions in $\mathcal{T}$, i.e. $C^{\mathcal{I}}{\subseteq}D^{\mathcal{I}}$ for every $C{\sqsubseteq}D \in \mathcal{T}$.

Constraint Box. A *constraint box* (CBox, for short) consists of a TBox $\mathcal{T}$ and a set RI of role inclusions of the form $r_1 \circ \cdots \circ r_n \sqsubseteq s$. (We will view CBoxes as unions $GCI \cup RI$ of general concept inclusions (GCI) and role inclusions (RI).) An interpretation $\mathcal{I}$ is a model of the CBox $\mathcal{C} = GCI \cup RI$ if it is a model of GCI and satisfies all role inclusions in $\mathcal{C}$, i.e. $r_1^{\mathcal{I}} \circ \ldots \circ r_n^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ for all $r_1 \circ \ldots \circ r_n \subseteq s \in RI$.

Definition 1. *Let C_1, C_2 be two concept descriptions.*

- *If $\mathcal{T}$ is a TBox, we say that C_1 is subsumed by C_2 w.r.t. $\mathcal{T}$ (denoted $C_1 \sqsubseteq_{\mathcal{T}} C_2$) iff $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.*
- *If $\mathcal{C}$ is a CBox, then $C_1 \sqsubseteq_{\mathcal{C}} C_2$ iff $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{C}$.*

The simplest propositionally closed description logic is $\mathcal{ALC}$ which allows for conjunction, disjunction, negation and existential and universal role restrictions. For description logics that allow full negation, subsumption tests w.r.t. TBoxes or CBoxes are reducible to satisfiability testing for concepts (i.e. checking if there exists a model of the TBox resp. CBox for which the interpretation of the concept is non-empty). It is well-known that for $\mathcal{ALC}$ subsumption checking (w.r.t. TBoxes and CBoxes) is in EXPTIME (cf. [5]). For *lightweight description logics* which do not allow negation, things are different: The main reasoning task is subsumption testing, which is the problem we consider in this paper.

We now define the fragments of the description logics $\mathcal{FL}_0$ used in this paper as well as the description logic $\mathcal{EL}$. [3]

The Description Logic $\mathcal{FL}_0$. $\mathcal{FL}_0$ is a lightweight description logic that only allows as concept constructors conjunction, universal role restrictions, and top concept. The subsumption problem w.r.t. general TBoxes is known to be in EXPTIME [4]. Fragments of $\mathcal{FL}_0$ resp. specific classes of subsumption for which the complexity is known to be lower include:

- **Subsumption w.r.t. standard TBoxes** has PSPACE complexity [8].
- **Subsumption w.r.t. acyclic TBoxes** is co-NP complete (where an acyclic TBox is a standard TBox that does not contain concept definitions $A_1 \equiv C_1, \ldots, A_k \equiv C_k$ such that $A_{i+1 \bmod k}$ is used in C_i for all $i < k$ [10]).
- **Horn-$\mathcal{FL}_0^+$** [9] is a variant of $\mathcal{FL}_0$ that both extends and restricts its expressivity in such a way that the subsumption problem remains in PTIME. It restricts $\mathcal{FL}_0$ axioms to the form shown in Table 2. The form of the axioms is limited in such a way that they can be rewritten into 3-variable function-free Horn-logic. It follows from this correspondence that verifying consistency of a Horn-$\mathcal{FL}_0^+$ knowledge base can be done in polynomial time. A Horn-$\mathcal{FL}_0$ TBox (CBox) consists only of inclusions of the form indicated in the first two lines of Table 2.

The Description Logic $\mathcal{EL}^+$. The description logic $\mathcal{EL}$ [1] allows as concept constructors only conjunction, existential role restrictions, and the bottom concept. $\mathcal{EL}^+$ [2, 4, 3] additionally allows for nominals and role composition. For $\mathcal{EL}^+$, checking CBox subsumption can be done in PTIME [4, 2].

[3] For the sake of simplicity, everywhere in what follows we consider fragments of these logics without nominals and without ABoxes.

$A \sqsubseteq C$	$\top \sqsubseteq C$	$R \sqsubseteq T$	$A \sqsubseteq \forall R.C$
$A \sqcap B \sqsubseteq C$	$A \sqsubseteq \bot$	$R \circ S \sqsubseteq T$	
$R(i, j)$	$A(i)$	$i \approx j$	

Table 2. Normal form for Horn $\mathcal{FL}_0^+$. A, B, C are names of atomic concepts; R, S, T are (possibly inverse) role names.

2.1 Combining $\mathcal{FL}_0$ and $\mathcal{EL}$

Let N_C be a set of concept names, and $N_R, N_{R'}$ be disjoint sets of role names. We propose a combination of $\mathcal{EL}$ (with roles in N_R) and $\mathcal{FL}_0$ (with roles in $N_{R'}$). The problem we study for such combinations is subsumption between concept expressions using constructs from both logics (such that existential restriction is used only for roles in N_R and universal restriction only for roles in $N_{R'}$) w.r.t. mixed TBoxes, consisting of an $\mathcal{EL}$ part and an $\mathcal{FL}_0$ part. We allow these TBoxes to share concept names (but the role names used in each type of axioms have to be disjoint). We have to impose the restriction that $N_R \cap N_{R'} = \emptyset$ in order to be sure that fine-grained complexity results can be obtained for TBox subsumption in such combinations, since the description logic combining these features freely, $\mathcal{ALEU}$, has an EXPTIME complexity for the subsumption problem w.r.t. TBox[4].

Definition 2. *A* mixed TBox *is a TBox $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_F$ which consists of two distinct parts: A set $\mathcal{T}_E$ of $\mathcal{EL}$ GCI (with role names N_R), and a set $\mathcal{T}_F$ of $\mathcal{FL}_0$ GCI (with role names $N_{R'}$), each respecting the syntactic restrictions imposed by their logic. In a* mixed TBox with acyclic $\mathcal{FL}_0$ part*, $\mathcal{T}_F$ is a standard acyclic TBox; in a* mixed TBox with standard $\mathcal{FL}_0$ part*, $\mathcal{T}_F$ is a standard TBox.*

We will use the names $\mathcal{EL}$-TBox and $\mathcal{FL}$-TBox to denote the set of $\mathcal{EL}$ (resp. Horn-$\mathcal{FL}_0$) inclusion axioms in a mixed TBox.

3 Algebraic Semantics

We assume known notions such as partially-ordered set, semilattice, lattice and Boolean algebra. For further information cf. [11]. We define a translation of concept descriptions into terms in a signature naturally associated with the set of constructors. For every role name r, we introduce unary function symbols, $f_{\exists r}, f_{\forall r}$. The renaming is inductively defined by:

- $\overline{C} = C$ for every concept name C;
- $\overline{\neg C} = \neg \overline{C}$; $\overline{C_1 \sqcap C_2} = \overline{C}_1 \wedge \overline{C}_2$, $\overline{C_1 \sqcup C_2} = \overline{C}_1 \vee \overline{C}_2$;
- $\overline{\exists r.C} = f_{\exists r}(\overline{C})$, $\overline{\forall r.C} = f_{\forall r}(\overline{C})$.

There exists a one-to-one correspondence between interpretations $\mathcal{I} = (D, \cdot^{\mathcal{I}})$ and Boolean algebras of sets $(\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\exists r}, f_{\forall r}\}_{r \in N_R})$, together with

[4] This follows from the fact that $\mathcal{ALEU}$ can simulate $\mathcal{ALC}$ [7].

valuations $v : N_C \to \mathcal{P}(D)$, where $f_{\exists r}, f_{\forall r}$ are defined, for every $U \subseteq D$, by:

$$f_{\exists r}(U) = \{x \mid \exists y((x, y) \in r^{\mathcal{I}} \text{ and } y \in U)\}$$
$$f_{\forall r}(U) = \{x \mid \forall y((x, y) \in r^{\mathcal{I}} \Rightarrow y \in U)\}.$$

Consider the following classes of algebras:

- BAO_{N_R}, the class of all Boolean algebras with operators $(B, \vee, \wedge, \neg, 0, 1, \{f_{\exists r}, f_{\forall r}\}_{r \in N_R})$, where
  - $f_{\exists r}$ is a join-hemimorphism, i.e. $f_{\exists r}(x \vee y) = f_{\exists r}(x) \vee f_{\exists r}(y)$, $f_{\exists r}(0) = 0$;
  - $f_{\forall r}$ is a meet-hemimorphism, i.e. $f_{\forall r}(x \wedge y) = f_{\forall r}(x) \wedge f_{\forall r}(y)$, $f_{\forall r}(1) = 1$;
  - $f_{\forall r}(x) = \neg f_{\exists r}(\neg x)$ for every $x \in B$.
- $\mathsf{BAO}^{\exists}_{N_R}$ the class of boolean algebras with operators $(B, \vee, \wedge, \neg, 0, 1, \{f_{\exists r}\}_{r \in N_R})$, such that $f_{\exists r}$ is a join-hemimorphism.
- $\mathsf{BAO}^{\forall}_{N_{R'}}$ the class of boolean algebras with operators $(B, \vee, \wedge, \neg, 0, 1, \{f_{\forall r}\}_{r \in N_{R'}})$, such that $f_{\forall r}$ is a meet-hemimorphism.
- $\mathsf{SLO}^{\exists}_{N_R}$ the class of all $\wedge$-semilattices with operators $(S, \wedge, 0, 1, \{f_{\exists r}\}_{r \in N_R})$, such that $f_{\exists r}$ is monotone and $f_{\exists r}(0) = 0$.
- $\mathsf{SLO}^{\forall}_{N_{R'}}$ the class of all $\wedge$-semilattices with operators $(S, \wedge, 0, 1, \{f_{\forall r}\}_{r \in N_{R'}})$, such that $f_{\forall r}$ is a meet-hemimorphism and $f_{\forall r}(1) = 1$.
- $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}}$ the class of all $\wedge$-semilattices with operators $(S, \wedge, 0, 1, \{f_{\exists r}\}_{r \in N_R}, \{f_{\forall r}\}_{r \in N_{R'}})$, such that $f_{\exists r}$ is monotone and $f_{\exists r}(0) = 0$, and $f_{\forall r}$ is a meet-hemimorphism and $f_{\forall r}(1) = 1$.

It is known that the TBox subsumption problem for $\mathcal{ALC}$ can be expressed as a uniform word problem for Boolean algebras with suitable operators (cf. e.g. [6]).

Let RI, RI' be sets of axioms of the form $r \sqsubseteq s$ and $r_1 \circ r_2 \sqsubseteq r$, with $r, s, r_1, r_2 \in N_R$ (resp. $r, s, r_1, r_2 \in N_{R'}$). We associate with RI, RI' the following set of axioms:

$$\begin{aligned} RI_a = \{&\forall x \;\; (f_{\exists r_2} \circ f_{\exists r_1})(x) \leq f_{\exists r}(x) \mid r_1 \circ r_2 \sqsubseteq r \in RI\} \cup \\ \{&\forall x \;\; f_{\exists r}(x) \leq f_{\exists s}(x) \mid r \sqsubseteq s \in RI\} \\ RI'_a = \{&\forall x \;\; (f_{\forall r_2} \circ f_{\forall r_1})(x) \geq f_{\forall r}(x) \mid r_1 \circ r_2 \sqsubseteq r \in RI'\} \cup \\ \{&\forall x \;\; f_{\forall r}(x) \geq f_{\forall s}(x) \mid r \sqsubseteq s \in RI'\} \end{aligned}$$

where $f \circ g$ denotes the composition of the functions f, g. Let $\mathsf{BAO}^{\exists}_{N_R}(RI)$ (resp. $\mathsf{SLO}^{\exists}_{N_R}(RI)$) be the subclass of $\mathsf{BAO}^{\exists}_{N_R}$ ($\mathsf{SLO}^{\exists}_{N_R}$) consisting of those algebras which satisfy RI_a, and $\mathsf{BAO}^{\forall}_{N_{R'}}(RI')$ (resp. $\mathsf{SLO}^{\forall}_{N_{R'}}(RI')$) be the subclass of $\mathsf{BAO}^{\forall}_{N_{R'}}$ ($\mathsf{SLO}^{\forall}_{N_{R'}}$) consisting of the algebras satisfying RI'_a.

In [13] we studied the link between TBox subsumption in $\mathcal{EL}$ and uniform word problems in the corresponding classes of semilattices with monotone functions, and in [14] we studied an extension to $\mathcal{EL}^+$. We will present these results here, together with an algebraic semantics for $\mathcal{FL}_0$.

Theorem 1 ([13]) *Assume that the only concept constructors are intersection and existential restriction. Then for all concept descriptions D_1, D_2 and every $\mathcal{EL}^+$ CBox $\mathcal{C} = GCI \cup RI$, with concept names $N_C = \{C_1, \dots, C_n\}$:*

$D_1 \sqsubseteq_{\mathcal{C}} D_2$ *iff* $\mathsf{SLO}^{\exists}_{N_R}(RI) \models \forall C_1 \dots C_n((\bigwedge_{C \sqsubseteq D \in GCI} \overline{C} \leq \overline{D}) \to \overline{D_1} \leq \overline{D_2})$.

We give a similar result for $\mathcal{FL}_0^+$.

Theorem 2 *Assume that the only concept constructors are intersection and universal restriction. Then for all concept descriptions D_1, D_2 and every $\mathcal{FL}_0^+$ CBox $\mathcal{C}$=GCI$\cup$RI, with concept names $N_C = \{C_1, \dots, C_n\}$:*

$$D_1 \sqsubseteq_{\mathcal{C}} D_2 \quad \textit{iff} \quad \mathsf{SLO}^{\forall}_{N_{R'}}(RI) \models \forall C_1 \dots C_n ((\bigwedge_{C \sqsubseteq D \in GCI} \overline{C} \leq \overline{D}) \rightarrow \overline{D_1} \leq \overline{D_2}).$$

3.1 Algebraic Semantics for a Combination of $\mathcal{EL}$ and $\mathcal{FL}_0$

Theorem 3 *Assume the only concept constructors are intersection, existential restriction over roles in N_R and universal restriction over roles in $N_{R'}$. Let $\mathcal{T}$ be a mixed TBox consisting of an $\mathcal{EL}$-TBox $\mathcal{T}_E$ (with roles in N_R) and an $\mathcal{FL}_0$-TBox $\mathcal{T}_F$ (with roles in $N_{R'}$), where $N_R \cap N_{R'} = \emptyset$. Then for all concept descriptions D_1, D_2 in the combined language, with concept names $N_C = \{C_1, \dots, C_n\}$:*

$$D_1 \sqsubseteq_{\mathcal{T}} D_2 \quad \textit{iff} \quad \mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}} \models \forall C_1 \dots C_n ((\bigwedge_{C \sqsubseteq D \in \mathcal{T}} \overline{C} \leq \overline{D}) \rightarrow \overline{D_1} \leq \overline{D_2}).$$

Note: The results can be extended in a natural way to $\mathcal{EL}^+$, $\mathcal{FL}_0^+$ and CBoxes (we will then take the combination of the role inclusions RI, RI', and the corresponding subclass $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}}(RI, RI')$ satisfying the axioms $RI_a \cup RI'_a$).

In what follows we show that we can reduce, in polynomial time and with a polynomial increase in the length of the formulae, the validity tasks w.r.t. $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}}$ to satisfiability tasks w.r.t. $\mathsf{SLO}^{\forall}_{N_{R'}}$ which can in general be solved in EXPTIME. We obtain the following finer grained results:

- If $\mathcal{T}_F$ is a standard TBox, the subsumption tasks are in PSPACE;
- If $\mathcal{T}_F$ is in the Horn-$\mathcal{FL}_0$ fragment, the reduction generates formulae whose satisfiability can be checked in PTIME.

For obtaining these results, we use the notion of local theory extensions, which is briefly introduced in what follows.

4 Local Theories and Local Theory Extensions

We here consider theories specified by their sets of axioms, and extensions of theories, in which the signature is extended by new *function symbols*. Let $\mathcal{T}_0$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, where Σ_0 a set of function symbols, and Pred a set of predicate symbols. We consider extensions $\mathcal{T}_1$ of $\mathcal{T}_0$ with signature $\Pi = (\Sigma, \mathsf{Pred})$, where $\Sigma = \Sigma_0 \cup \Sigma_1$ (i.e. the signature is extended by new function symbols). We assume that $\mathcal{T}_1$ is obtained from $\mathcal{T}_0$ by adding a set $\mathcal{K}$ of (universally quantified) clauses in the signature Π, each of them containing at least a function symbol in Σ_1 and denote this by writing $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$.

Locality. Let $\mathcal{K}$ be a set of (universally quantified) clauses in the signature Π. In what follows, when referring to sets G of ground clauses we assume they are in the signature $\Pi^c = (\Sigma \cup \Sigma_c, \mathsf{Pred})$ where Σ_c is a set of new constants. An extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is *local* if satisfiability of a set G of clauses w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ only depends on $\mathcal{T}_0$ and those instances $\mathcal{K}[G]$ of $\mathcal{K}$ in which the terms starting with extension functions are in the set $\mathsf{st}(\mathcal{K}, G)$ of ground terms which already occur in G or $\mathcal{K}$, i.e. if condition (Loc) is satisfied:

(Loc) For every finite set G of ground clauses $\mathcal{T}_1 \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models \perp$

where $\mathcal{K}[G] = \{C\sigma \mid C \in \mathcal{K},$ for each subterm $f(t)$ of C, with $f \in \Sigma_1$, $f(t)\sigma \in \mathsf{st}(\mathcal{K}, G)$, and for each variable x which does not occur below a function symbol in $\Sigma_1, \sigma(x) = x\}$.

Hierarchical Reasoning. In local theory extensions hierarchical reasoning is possible. All clauses in $\mathcal{K}[G] \cup G$ have the property that the function symbols in Σ_1 have as arguments only ground terms. Therefore, $\mathcal{K}[G] \cup G$ can be purified (i.e. the function symbols in Σ_1 are separated from the other symbols) by introducing, in a bottom-up manner, new constants c_t for subterms $t = f(g_1, \dots, g_n)$ with $f \in \Sigma_1$, g_i ground $\Sigma_0 \cup \Sigma_c$-terms (where Σ_c is a set of constants which contains the constants introduced by flattening, resp. purification), together with corresponding definitions $c_t \approx t$. The set of clauses thus obtained has the form $\mathcal{K}_0 \cup G_0 \cup D$, where D is a set of ground unit clauses of the form $f(g_1, \dots, g_n) \approx c$, where $f \in \Sigma_1$, c is a constant, $g_1, \dots, g_n$ are ground terms without function symbols in Σ_1, and $\mathcal{K}_0$ and G_0 are clauses without function symbols in Σ_1.

For the sake of simplicity in what follows we will always first flatten and then purify $\mathcal{K}[G] \cup G$. Thus we ensure that D consists of ground unit clauses of the form $f(c_1, \dots, c_n) \approx c$, where $f \in \Sigma_1$, and $c_1, \dots, c_n, c$ are constants.

Theorem 4 ([12]) *Let $\mathcal{K}$ be a set of clauses. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a local theory extension. For any set G of ground clauses, let $\mathcal{K}_0 \cup G_0 \cup D$ be obtained from $\mathcal{K}[G] \cup G$ by flattening and purification, as explained above. Then the following are equivalent:*

(1) $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$.
(2) $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models \perp$.
(3) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup N_0 \models \perp$, *where*

$$N_0 = \{\bigwedge_{i=1}^{n} c_i \approx d_i \rightarrow c \approx d \mid f(c_1, \dots, c_n) \approx c, f(d_1, \dots, d_n) \approx d \in D\}.$$

Theorem 5 ([15]) *The extension of any semilattice-ordered theory with monotone functions is local. In particular, the extension $\mathsf{SLO}^{\forall}_{N_{R'}} \subseteq \mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}}$ of the theory of semilattices with meet-hemimorphisms in a set $\{f_{\forall R} \mid R \in N_{R'}\}$ with monotone functions in a set $\{f_{\exists R} \mid R \in N_R\}$, where $N_R \cap N_{R'} = \emptyset$, is local.*

Thus, the method for hierarchical reasoning described in Theorem 4 can be used in this context to reduce the proof tasks in $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}}$ to proof tasks in $\mathsf{SLO}^{\forall}_{N_{R'}}$. We describe the approach in the next section. For the sake of simplicity, in what follows we use the notation $\exists R.C$ for $f_{\exists R}(C)$ and $\forall S.D$ for $f_{\forall S}(D)$. [5]

[5] In [14] we proved generalized locality results also for extensions with monotone functions satisfying axioms of the form RI_a, so the results can be further extended to give a reduction of proof tasks in $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}}(RI, RI')$ to proof tasks in $\mathsf{SLO}^{\forall}_{N_{R'}}(RI')$.

5 The Combination of $\mathcal{EL}$ and $\mathcal{FL}_0$

We consider the subsumption problem for the combination of $\mathcal{EL}$ and $\mathcal{FL}_0$ introduced in Section 3.1 and illustrate the way hierarchical reasoning can be used for reasoning in this combination, and for identifying fragments of this combination and subsumption tasks which can be checked in PSPACE/PTIME. [6]

We first have to *purify* the expressions for which we want to verify subsumption. Consider for instance the subsumption $C \sqsubseteq \exists R.D$, where C and D are resp. an $\mathcal{FL}_0$ and an $\mathcal{EL}$ concept description. To purify it, we add the axiom $D' \equiv \exists R.D$ to the $\mathcal{EL}$-TBox (where D' is a new concept name) and rewrite the subsumption as $C \sqsubseteq D'$. We apply this process in an "inside-out" fashion such that the final result is checking subsumption between concept names w.r.t. to an augmented TBox. This procedure does not affect complexity when we use new names for $\mathcal{EL}$ concept descriptions ($\mathcal{EL}$ allows for equalities and inequalities TBoxes). In what follows, $C[\exists R.C']$ is a notation indicating that C is a concept description in the combination of $\mathcal{EL}$ and $\mathcal{FL}_0$ containing a subterm of the form $\exists R.C'$, $R \in N_R$; the notation $C[C'']$ indicates the concept description obtained by replacing $\exists R.C'$ with C'' in C.

Theorem 6 *Consider the subsumption problem $C[\exists R.C'] \sqsubseteq_{\mathcal{T}} D$ (where C' is an $\mathcal{EL}$ concept description) w.r.t. a mixed TBox $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_F$ and the subsumption problem $C[C''] \sqsubseteq_{\mathcal{T}'} D$ w.r.t. the extension $\mathcal{T}'$ of $\mathcal{T}$ with a new concept name C'' together with its definition $C'' \equiv \exists R.C'$. Then the following are equivalent:*

(1) $\mathsf{SLO}^{\exists\forall}_{N_R,N_{R'}} \models (\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}_E \cup \mathcal{T}_F} C_1 \leq C_2) \rightarrow C[\exists R.C'] \leq D$
(2) $\mathsf{SLO}^{\exists\forall}_{N_R,N_{R'}} \models (\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}_E \cup \mathcal{T}_F} C_1 \leq C_2 \quad \wedge \quad C'' \approx \exists R.C') \rightarrow C[C''] \leq D$

This also holds for subsumption problems of the form $C \sqsubseteq D[\exists R.D']$.

Theorem 7 *Consider the subsumption problem $C[\forall S.C'] \sqsubseteq_{\mathcal{T}} D$ (where C' is an $\mathcal{FL}_0$ concept description) w.r.t. a mixed TBox $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_F$ and the subsumption problem $C[C''] \sqsubseteq_{\mathcal{T}'} D$ w.r.t. the extension $\mathcal{T}'$ of $\mathcal{T}$ with a new concept name C'' and a definition for it ($C'' \equiv \forall S.C'$). Then the following are equivalent:*

(1) $\mathsf{SLO}^{\exists\forall}_{N_R,N_{R'}} \models (\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}_E \cup \mathcal{T}_F} C_1 \leq C_2) \rightarrow C[\forall S.C'] \leq D$
(2) $\mathsf{SLO}^{\exists\forall}_{N_R,N_{R'}} \models (\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}_E \cup \mathcal{T}_F} C_1 \leq C_2 \ \wedge \ C'' \approx \forall S.C') \rightarrow C[C''] \leq D.$

This also holds for subsumption problems of the form $C \sqsubseteq D[\forall S.D']$.

$\mathcal{FL}_0$ with Standard TBoxes. Assume that we consider a combination of $\mathcal{EL}$ with the fragment of $\mathcal{FL}_0$ with standard TBoxes. Then $\mathcal{T}_F$ is a standard $\mathcal{FL}_0$-TBox, hence also $\mathcal{T}_F \cup \{C'' \equiv \forall S.C'\}$ is a standard TBox.

$\mathcal{FL}_0$ with Acyclic TBoxes. Assume that we consider a combination of $\mathcal{EL}$ with the fragment of $\mathcal{FL}_0$ with acyclic standard TBoxes, i.e. $\mathcal{T}_F$ is a standard

[6] The results can be extended to combinations of $\mathcal{EL}^+$ and $\mathcal{FL}_0^+$ and to subsumption tasks w.r.t. CBoxes. Due to space constraints this extension is not presented here.

acyclic TBox $\{A_i \equiv C_i \mid i = 1, ..., k\}$. Assume that C' does not contain any of the atomic concept names A_i. Since C'' is a new concept name, the $\mathcal{FL}_0$-TBox $\mathcal{T}_F \cup \{C'' \equiv \forall S.C'\}$ is an acyclic TBox. After the elimination of $\exists R.C$ concepts and introduction of new concept names and definitions, the resulting TBox is a standard $\mathcal{FL}_0$-TBox (which is acyclic only if additional acyclicity assumptions are made on $\mathcal{T}_E$).

Horn-$\mathcal{FL}_0$. The restriction imposed on the form of the TBox axioms in Horn-$\mathcal{FL}_0$ prevents purification by adding definitions of the form $C'' \equiv \forall S.C'$ (we cannot allow universal restriction on the left-hand side of an axiom). For the case where we have to purify the left-hand side that causes no problem since if $\forall S.C'$ occurs on the left-hand side we only need to add $C'' \sqsubseteq \forall S.C'$ to the TBox:

Theorem 8 *Consider the subsumption problem $C[\forall S.C'] \sqsubseteq_{\mathcal{T}} D$ (where C' is an $\mathcal{FL}_0$ concept description) w.r.t. a mixed TBox $\mathcal{T} = \mathcal{T}_E \cup \mathcal{T}_F$, and the subsumption problem $C[C''] \sqsubseteq_{\mathcal{T}'} D$ w.r.t. the extension $\mathcal{T}'$ of $\mathcal{T}$ with a new concept name C'' and an inclusion of the form $(C'' \sqsubseteq \forall S.C')$. Then the following are equivalent:*

(1) $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}} \models (\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}_E \cup \mathcal{T}_F} C_1 \leq C_2) \rightarrow C[\forall S.C'] \leq D.$
(2) $\mathsf{SLO}^{\exists\forall}_{N_R, N_{R'}} \models (\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}_E \cup \mathcal{T}_F} C_1 \leq C_2 \ \wedge \ C'' \leq \forall S.C') \rightarrow C[C''] \leq D.$

However, we cannot replace universal restriction on the right-hand side with a name in general which prevents us to purify arbitrary expressions.

Hierarchical Reasoning. Consider the purified form of the problem. We replace all terms of the form $\exists R.C$ in $\mathcal{T}_E$ with a new constant, say $C_{\exists R.C}$. Let Def be the set of all definitions for these new constants, of the form $C_{\exists R.C} \equiv \exists R.C$. Let M_0 be the set of corresponding instances of monotonicity axioms:

$$M_0 = \{C_1 \leq C_2 \rightarrow C_{\exists R.C_1} \leq C_{\exists R.C_2} \mid C_{\exists R.C_i} = \exists R.C_i \in \mathsf{Def}\}.$$

Let $(\mathcal{T}_E)_0$ be the purified form of $\mathcal{T}_E$. By Theorem 4, the following are equivalent:

(i) $\mathsf{SLO}^{\forall\exists}_{N_R, N_{R'}} \models \bigwedge_{(D \sqsubseteq D') \in \mathcal{T}} D \leq D' \rightarrow C_1 \leq C_2$.
(ii) $G_0 \wedge M_0$ is unsatisfiable in $\mathsf{SLO}^{\forall}_{N_{R'}}$, where $G_0 = (\mathcal{T}_E)_0 \wedge \mathcal{T}_F \wedge (\neg(C_1 \leq C_2))_0$.

Note that in the presence of the monotonicity axioms, the instances of the congruence axioms in N_0 (cf. notation in Theorem 4) are redundant.

Theorem 9 *Assume that the only concept constructors are intersection and existential restrictions over roles in N_R and universal restrictions over roles in $N_{R'}$. Assume that we have a mixed TBox, consisting of an $\mathcal{EL}$-TBox $\mathcal{T}_E$ (with roles in a set N_R) and an $\mathcal{FL}_0$-TBox $\mathcal{T}_F$ (with roles in a set $N_{R'}$), where $N_R \cap N_{R'} = \emptyset$. Then for all concept descriptions D_1, D_2 with concept names $N_C = \{C_1, \dots, C_n\}$ over this signature, the following hold:*

(1) If $\mathcal{T}_F$ is a standard TBox, then:
  (a) For any subsumption problem purification yields a new mixed TBox $\mathcal{T}' = \mathcal{T}'_E \cup \mathcal{T}'_F = \mathcal{T}_E \wedge \mathsf{Def} \wedge \mathcal{T}_F$ with a standard $\mathcal{FL}_0$ part, and after the elimination of $\exists R.C$ concepts, $(\mathcal{T}'_E)_0 \cup \mathcal{T}'_F$ is a standard $\mathcal{FL}_0$ TBox.

(b) *Checking whether* $D_1 \sqsubseteq_{\mathcal{T}_E \cup \mathcal{T}_F} D_2$ *can be done in PSPACE.*

(2) *If* $\mathcal{T}_F$ *is a Horn-*$\mathcal{FL}_0$ *TBox and* C *is an arbitrary concept description in the combined language and* D *does not contain terms of the form* $\exists R.D_1$*, where* $R \in N_R$ *with subterms of the form* $\forall S.D_2$*,* $S \in N_{R'}$*, then:*

(a) *Purification yields a new mixed TBox with a Horn-*$\mathcal{FL}_0$ *part; after the elimination of* $\exists R.C$ *concepts,* $(\mathcal{T}'_E)_0 \cup \mathcal{T}'_F$ *is a Horn-*$\mathcal{FL}_0$ *TBox. Since*
(i) $C \sqsubseteq_{\mathcal{T}} D_1 \sqcap D_2$ *iff* ($C \sqsubseteq_{\mathcal{T}} D_1$ *and* $C \sqsubseteq_{\mathcal{T}} D_2$)*, and*
(ii) $\forall S$ *commutes with intersections,*
we can consider, w.l.o.g. only subsumption problems $D_1 \sqsubseteq_{\mathcal{T}} \forall S_1. \dots \forall S_n.D$*,* $n \geq 0$*, where* D_2, D *are concept names.*

(b) *Checking whether* $D_1 \sqsubseteq_{\mathcal{T}_E \cup \mathcal{T}_F} D_2$ *where* $D_2 = \forall S_1. \dots \forall S_n.D$ *(where* $n \geq 0$ *and* C, D *are concept names) can be done in PTIME.*

Proof. (1)(a) and (2)(a) are simple consequences of the purification procedure. Consider the purified form of the problem By Theorems 3 and 4, $D_1 \sqsubseteq_{\mathcal{T}_E \cup \mathcal{T}_F} D_2$ iff $\mathsf{SLO}^{\forall\exists}_{N_R, N_{R'}} \models \bigwedge_{D \sqsubseteq D' \in \mathcal{T}} (D \leq D' \rightarrow D_1 \leq D_2$ iff $G_0 \wedge M_0$ is unsatisfiable in $\mathsf{SLO}^{\forall}_{N_{R'}}$, where $G_0 = (\mathcal{T}_E)_0 \wedge \mathcal{T}_F \wedge (\neg(C_1 \leq C_2))_0$. In order to test the unsatisfiability of the latter problem we proceed as follows. We first note that, due to the convexity of $\mathsf{SLO}^{\forall}_{N_{R'}}$, if $G_0 \wedge M_0 \models \perp$, then there exists a clause $C = (c_1 \leq d_1 \rightarrow c \leq d) \in M_0$ such that $G_0 \models c_1 \leq d_1$ and $G_0 \wedge \{c \leq d\} \wedge M_0 \backslash \{C\} \models \perp$. By iterating the argument above we can always successively entail sufficiently many premises of monotonicity axioms in order to ensure that there exists a set $\{C_1, \dots, C_n\}$ of clauses in M_0 with $C_j = (c^j_1 \leq d^j_1 \rightarrow c^j \leq d^j)$, such that for all $k \in \{0, \dots, n-1\}$, $G_0 \wedge \bigwedge_{j=1}^k (c^j \leq d^j) \models \bigwedge c^{k+1}_i \leq d^{k+1}_i$ and $G_0 \wedge \bigwedge_{j=1}^n (c^j \leq d^j) \models \perp$. Conversely, if the last condition holds, then $G_0 \wedge M_0 \models \perp$. This means that in order to test satisfiability of $G_0 \wedge M_0$ we need to: (i) test entailment of the premises of M_0 from G_0; when all premises of some clause are provably true we delete the clause and add its conclusion to G_0, and (ii) in the end check whether $G_0 \wedge \bigwedge_{j=1}^n (c^j \leq d^j) \models \perp$.

Under the assumptions in (1), every entailment task in (i) and the test in (ii) are in PSPACE. Since space can be reused, the process terminates and is in PSPACE. Under the assumptions in (2), $\mathcal{T}_0 = (\mathcal{T}_E)_0 \cup \mathcal{T}_F$ and G_0 are in Horn $\mathcal{FL}_0$. Therefore, every entailent task in (i) above can be done in PTIME. The task (ii) - for the case that G_0 is derived from a subsumption problem of the form $C \sqsubseteq_{\mathcal{T}} \forall S_1. \dots \forall S_n.D$, where $n \geq 0$, and C, D are concept names, can be translated to a satisfiability test in Horn-$\mathcal{FL}_0$, so it can be done in PTIME. □

6 Conclusion

We identified a class of subsumption problems in a combination of $\mathcal{EL}$ and Horn-$\mathcal{FL}_0$, which can be checked in PTIME. Since $\mathcal{FL}_0$ allows universal role restriction and $\mathcal{EL}$ allows existential role restrictions, we thus have a framework where subsumption between expressions including both types of role restrictions (but for disjoint sets of roles) can be checked in polynomial space or time.

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