Finite Model Reasoning in *DL-Lite* with Cardinality Constraints*
(Preliminary Results)

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1 Introduction

The relationship of description logics (DLs) and conceptual modelling has been extensively studied in the literature [5, 4, 1]. One of the advantages of using description logics as modelling languages is that along with their capability of representing knowledge they provide also reasoning services. More precisely, a conceptual model can be represented by a DL ontology (TBox), and standard reasoning services (e.g., satisfiability and subsumption) allow to verify some properties of the conceptual model (e.g., consistency) and infer relations between concepts (IS-A relationships between classes or entities) that are not explicitly expressed. In order to use DLs effectively for conceptual modelling we need to ensure (1) that the chosen DL language is expressive enough to capture faithfully the intended semantics of traditional modelling languages (e.g., UML class diagrams, ER schema), and (2) that the complexity of reasoning in the chosen DL is acceptable (e.g., tractable). Regarding (1), it is worth noticing that the domain of interest in most applications is finite, therefore, reasoning on conceptual models should be understood as *reasoning w.r.t. finite models*. The latter is not the usual assumption in DLs mainly because traditional description logics enjoy the finite model property (FMP), and hence there is no need to distinguish between reasoning w.r.t. arbitrary models, and w.r.t. finite ones. Notably, $\mathcal{ALC}$ (one of the traditional DLs) is not expressive enough for capturing *cardinality constraints*. In DLs cardinality constraints are expressed by (qualified) number restrictions. $\mathcal{ALCQI}$ –which extends $\mathcal{ALC}$ with qualified number restrictions and inverse roles– captures the semantics of UML class diagrams [4]. However, this extension of $\mathcal{ALC}$ does not enjoy the FMP any more. One drawback for the use of $\mathcal{ALCQI}$ is the complexity of reasoning: finite satisfiability of $\mathcal{ALCQI}$ knowledge bases is ExpTime-complete [11]. The high complexity of reasoning makes $\mathcal{ALCQI}$ not very attractive for the conceptual modelling task; specially because no *optimized algorithms* for finite model reasoning exist. As an alternative, members of the *DL-Lite*-family of description logics including unqualified

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number restrictions [2] capture relevant modelling features [1]. However, little has been done in the study of the complexity of reasoning w.r.t. finite models in *DL-Lite* [13]. A consideration around the use of number restrictions (qualified or unqualified) regards their semantics: on the DL side, number restrictions, intended for possible infinite model semantics, constraint the numbers of objects that are related to a certain object; while cardinality constraints in conceptual modelling, intended for finite model semantics, establish relationships among the cardinality of classes/entities.

The purpose of this paper is to bring attention to finite model reasoning in description logics from a model theoretical view point. We adapt existing techniques [6] and show that the complexity of finite model reasoning in the Horn fragment of *DL-Lite* is tractable when only global functionality constraints are considered. While this result seems to be an almost straightforward consequence of existing results [13]; the approach taken in this paper leads to a deeper understanding of the structural properties of finite models for *DL-Lite* knowledge bases. We also observe that when allowing the use of arbitrary cardinality constraints, finite satisfiability becomes harder than arbitrary reasoning in $\textit{DL-Lite}^{\mathcal{N}}_{horn}$. In Section 5 we provide an intuition for an upper bound on the complexity of finite model reasoning in $\textit{DL-Lite}^{\mathcal{N}}_{horn}$. The results and observations presented in this paper shall serve as the foundation for future work on the finite model theory in light weight description logics [2, 3].

2 Preliminaries

***DL-Lite* syntax and semantics** The language of $\textit{DL-Lite}^{\mathcal{N}}_{horn}$ contains *individual names* $a_0, a_1, \ldots$, *concept names* $A_0, A_1, \ldots$, *role names* $P_0, P_1, \ldots$. *Complex roles* R, and *concepts* B are built according to the following syntax rule:

$$R \ ::= \ P_i \ \mid \ P_i^-, \qquad B \ ::= \ \bot \ \mid \ A_i \ \mid \ \geq n\ R,$$

where n in *number restrictions* ($\geq n\ R$) is a positive integer. We call *existentials* those number restrictions with $n = 1$, denoted also by $\exists R$. A $\textit{DL-Lite}^{\mathcal{N}}_{horn}$-TBox $\mathcal{T}$ is a finite set of axioms of the form $B_1 \sqcap \cdots \sqcap B_k \sqsubseteq B, \quad k \geq 0$, where by definition the empty conjunction is $\top$. We also consider the sublogic $\textit{DL-Lite}^{\mathcal{F}}_{horn}$, which of all number restrictions only allows for existentials, and those with $n = 2$ occurring only in concept inclusions of the form $\geq 2\ R \sqsubseteq \bot$, which are called *global functionality constraints*, and are denoted by ($\mathsf{funct}\ R$). An ABox $\mathcal{A}$ is a finite set of assertions of the form: $A(a_i)$ or $P(a_i, a_j)$. Together, a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$ constitute a $\textit{DL-Lite}^{\mathcal{F}}_{horn}$ *knowledge base*(KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. We use $\textit{ind}(\mathcal{A})$ to denote the set of individual names occurring in $\mathcal{A}$; $\textit{role}(\mathcal{K})$ the set of role names in $\mathcal{K}$, and $\textit{role}^{\pm}(\mathcal{K})$ the set of roles $\{P_k, P_k^- \mid P_k \in \textit{role}(\mathcal{K})\}$. For a role $R \in \textit{role}^{\pm}(\mathcal{K})$, $R^- = P_k$ if $R = P_k^-$, and $R^- = P_k^-$ if $R = P_k$. Finally, $\textit{concepts}(\mathcal{K})$ denotes the set of basic concepts occurring in $\mathcal{K}$, and $\textit{concepts}(\mathcal{T})$, for those occurring in $\mathcal{T}$.

An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty *domain* $\Delta^{\mathcal{I}}$, and an *interpretation function* $\cdot^{\mathcal{I}}$ that assigns to each individual name a_i an element

$a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$; to each concept name A_j a subset $A_j^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and to each role name P_k, a binary relation $P_k^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation function is extended to concepts and roles as follows:

$$\begin{aligned}
(P_k^-)^{\mathcal{I}} &= \{(e,d) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (d,e) \in P_k^{\mathcal{I}}\}; && \text{(inverse role)}\\
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}}; && \text{(Top)}\\
\bot^{\mathcal{I}} &= \emptyset; && \text{(Bottom)}\\
(\geq n\ R)^{\mathcal{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \sharp\{e \in \Delta^{\mathcal{I}} \mid (d,e) \in R^{\mathcal{I}}\} \geq n\}; && \text{(number rest.)}\\
(B_i \sqcap B_j)^{\mathcal{I}} &= B_i^{\mathcal{I}} \cap B_j^{\mathcal{I}}; && \text{(conjunction)}
\end{aligned}$$

where $\sharp$ denotes the cardinality of a set. An interpretation $\mathcal{I}$ *satisfies* a TBox axiom $\bigsqcap_k B_k \sqsubseteq B$ iff $(\bigsqcap_k B_k)^{\mathcal{I}} \subseteq B^{\mathcal{I}}$, in that case we write $\mathcal{I} \models \bigsqcap_k B_k \sqsubseteq B$; similarly, $\mathcal{I} \models (\mathsf{funct}\ R)$ iff whenever both $(d,e) \in R^{\mathcal{I}}$ and $(d,e') \in R^{\mathcal{I}}$, then $e = e'$. For ABox assertions we have that $\mathcal{I} \models A(a_i)$ iff $a_i^{\mathcal{I}} \in A^{\mathcal{I}}$; and $\mathcal{I} \models P_k(a_i, a_j)$ iff $(a_i^{\mathcal{I}}, a_j^{\mathcal{I}}) \in P_k^{\mathcal{I}}$. A knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is *satisfiable* (or consistent) if there is an interpretation $\mathcal{I}$, satisfying every axiom in $\mathcal{T}$ and every assertion in $\mathcal{A}$. In this case we write $\mathcal{I} \models \mathcal{K}$ (as well as $\mathcal{I} \models \mathcal{T}$, and $\mathcal{I} \models \mathcal{A}$), and we say that $\mathcal{I}$ is a *model* of $\mathcal{K}$ (and of $\mathcal{T}$ and $\mathcal{A}$). If $\mathcal{I}$ is finite (i.e., its domain is finite) we say that a $\mathcal{K}$ (as well as $\mathcal{T}$ and $\mathcal{A}$) is *finitely satisfiable.* The *type* of d in $\mathcal{I}$ is the set $t_{\mathcal{I}}(d) = \{B \mid d \in B^{\mathcal{I}}\}$, where B is a *DL-Lite*$_{horn}^{\mathcal{N}}$-concept. The set of all types of $\mathcal{I}$, is $types(\mathcal{I}) = \{t_{\mathcal{I}}(d) \mid d \in \Delta^{\mathcal{I}}\}$. We consider standard reasoning tasks. Specifically, *satisfiability* and *subsumption.* Let $\mathcal{L} \in \{$*DL-Lite*$_{horn}^{\mathcal{F}}$, *DL-Lite*$_{horn}^{\mathcal{N}}\}$. The *satisfiability problem* consists on deciding, given an $\mathcal{L}$-KB $\mathcal{K}$, whether $\mathcal{K}$ is satisfiable; while the *subsumption problem* amounts to decide, given an $\mathcal{L}$-TBox $\mathcal{T}$ and $\mathcal{L}$-concepts C_1 and C_2, whether $\mathcal{T} \models C_1 \sqsubseteq C_2$, i.e., whether $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ in every model $\mathcal{I}$ of $\mathcal{T}$.

3 Model Theoretical Characterizations

Lutz et. al., [10] provide a model theoretical characterization of *DL-Lite*$_{horn}$ (without number restrictions) based on *(equi)simulation*, a weaker notion of the classical (bi)simulation [7]. In order to capture the *counting* capability of *DL-Lite*$_{horn}^{\mathcal{N}}$ we extend this notion similarly to the graded-bisimulation in [12].

For a *DL-Lite*$_{horn}^{\mathcal{N}}$ interpretation $\mathcal{I}$, an object $d \in \Delta^{\mathcal{I}}$, and a role R,

$$R\text{-}\mathsf{succ}^{\mathcal{I}}(d) = \{e \in \Delta^{\mathcal{I}} \mid (d,e) \in R^{\mathcal{I}}\}$$

is the set of *R-successors* of d in $\mathcal{I}$.

Let $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ and $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$ be two *DL-Lite*$_{horn}^{\mathcal{N}}$ interpretations. A *graded equisimulation*(or *g*-equisimulation) between $\mathcal{I}_1$ and $\mathcal{I}_2$ is a relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ that satisfies the following properties:

(atom) for every concept name A, if $(d,e) \in \rho$ then $d \in A^{\mathcal{I}_1}$ iff $e \in A^{\mathcal{I}_2}$;
(role) for every role R, if $(d,e) \in \rho$, then the following hold:
  (i) for every finite set $S \subseteq R\text{-}\mathsf{succ}^{\mathcal{I}_1}(d)$, there exists a finite set $S' \subseteq R\text{-}\mathsf{succ}^{\mathcal{I}_2}(e)$, such that $\sharp S = \sharp S'$; and

(ii) for every finite set $S \subseteq R\text{-}\mathsf{succ}^{\mathcal{I}_2}(e)$, there exists a finite set $S' \subseteq R\text{-}\mathsf{succ}^{\mathcal{I}_1}(d)$, such that $\sharp S = \sharp S'$.[1]

ρ is called *global* if and only if (i) for every $d \in \Delta^{\mathcal{I}_1}$ there is some $e \in \Delta^{\mathcal{I}_2}$ with $(d, e) \in \rho$, (ii) for every $e \in \Delta^{\mathcal{I}_2}$ there is some $d \in \Delta^{\mathcal{I}_1}$ with $(d, e) \in \rho$.

We write $(\mathcal{I}_1, d) \approx (\mathcal{I}_2, e)$ if there exists a g-equisimulation ρ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $(d, e) \in \rho$. Finally, we say that $\mathcal{I}_1$ is *g-equisimilar* to $\mathcal{I}_2$, denoted as $\mathcal{I}_1 \approx \mathcal{I}_2$, if there is a *global g-equisimulation* ρ between $\mathcal{I}_1$ and $\mathcal{I}_2$.

Lemma 1. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{N}}_{horn}$ TBox, C a DL-Lite$^{\mathcal{N}}_{horn}$ concept; and $\mathcal{I}_1$ and $\mathcal{I}_2$ be two DL-Lite$^{\mathcal{N}}_{horn}$ interpretations over the signature of $\mathcal{T}$ and C. The following statements hold:*

(a) DL-Lite$^{\mathcal{N}}_{horn}$concepts are invariant under g-equisimulations: $(\mathcal{I}_1, d) \approx (\mathcal{I}_2, e)$ implies $d \in C^{\mathcal{I}_1}$ iff $e \in C^{\mathcal{I}_2}$.

(b) DL-Lite$^{\mathcal{N}}_{horn}$ TBoxes are invariant under global g-equisimulations: if $\mathcal{I}_1 \approx \mathcal{I}_2$ then $\mathcal{I}_1 \models \mathcal{T}$ iff $\mathcal{I}_2 \models \mathcal{T}$.

(c) Every model of a DL-Lite$^{\mathcal{N}}_{horn}$ TBox is g-equisimilar to a tree-shaped model.

Canonical Models We use a standard characterization of unrestricted entailment in terms of canonical models [8]. A canonical interpretation for a *DL-Lite*$^{\mathcal{N}}_{horn}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is constructed by (i) expanding the set of individual names in $\mathcal{A}$ with an additional set of individuals $\{d_R \mid R \in \mathit{role}^{\pm}(\mathcal{T})\}$ that serve as witness of existentials, and (ii) expanding the extensions of concept and role names as required by $\mathcal{T}$. A role R is called *generating* in $\mathcal{K}$ if there exist $a \in \mathit{ind}(\mathcal{A})$ and $R_0, \ldots, R_n = R$ such that the following conditions hold:

(agen) $\mathcal{K} \models \exists R_0(a)$ but $R_0(a, b) \notin \mathcal{A}$ for all $b \in \mathit{ind}(\mathcal{A})$ (written $a \rightsquigarrow d_{R_0^-}$).

(rgen) For $i < n, \mathcal{T} \models \exists R_i^- \sqsubseteq \exists R_{i+1}$ and $R_i^- \neq R_{i+1}$ (written $d_{R_i^-} \rightsquigarrow d_{R_{i+1}^-}$).

Definition 1. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a DL-Lite$^{\mathcal{N}}_{horn}$ KB. The canonical interpretation $\mathcal{I}_{\mathcal{K}} = (\Delta^{\mathcal{I}_{\mathcal{K}}}, \cdot^{\mathcal{I}_{\mathcal{K}}})$ of $\mathcal{K}$ is defined as follows:*

$$\begin{aligned}
\Delta^{\mathcal{I}_{\mathcal{K}}} &= \mathit{ind}(\mathcal{A}) \cup \{d_R \mid R^- \text{ is generating in } \mathcal{K}\};\\
a^{\mathcal{I}_{\mathcal{K}}} &= a \text{ for every } a \in \mathit{ind}(\mathcal{A});\\
A^{\mathcal{I}_{\mathcal{K}}} &= \{a \in \mathit{ind}(\mathcal{A}) \mid \mathcal{K} \models A(a)\} \cup \{d_R \in \Delta^{\mathcal{I}_{\mathcal{K}}} \mid \mathcal{T} \models \exists R \sqsubseteq A\};\\
P^{\mathcal{I}^{\mathcal{K}}} &= \{(a_i, a_j) \in \mathit{ind}(\mathcal{A}) \times \mathit{ind}(\mathcal{A}) \mid P(a_i, a_j) \in \mathcal{A}\} \cup\\
&\quad \{(a, d_{P^-}) \mid a \rightsquigarrow d_{P^-}\} \cup \{(d_P, a) \mid a \rightsquigarrow d_P\} \cup\\
&\quad \{(d_S, d_{P^-}) \mid d_S \rightsquigarrow d_{P^-}\} \cup \{(d_P, d_S) \mid d_S \rightsquigarrow d_P\}.
\end{aligned}$$

The canonical interpretation $\mathcal{I}_{\mathcal{K}}$ of a given KB $\mathcal{K}$ can be computed in polynomial time on the size of $\mathcal{K}$ [8], and serves as a finite compact representation of every model of $\mathcal{K}$. However, $\mathcal{I}_{\mathcal{K}}$ is *not* itself in general a model of $\mathcal{K}$, as the following example shows:

[1] Clearly, if both $R\text{-}\mathsf{succ}^{\mathcal{I}_1}(d)$ and $R\text{-}\mathsf{succ}^{\mathcal{I}_2}(e)$ are finite, these conditions are equivalent to $\sharp R\text{-}\mathsf{succ}^{\mathcal{I}_1}(d) = \sharp R\text{-}\mathsf{succ}^{\mathcal{I}_2}(e)$.

Example 1. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where $\mathcal{T} = \{\exists S \sqsubseteq \exists P_1, \exists P_1^- \sqsubseteq \exists P_2, \exists P_2^- \sqsubseteq \exists P_1, \exists P_2^- \sqsubseteq \exists P_3^-, \exists P_3 \sqsubseteq \exists S^-(\mathsf{funct}\ P_1^-), (\mathsf{funct}\ P_2^-), (\mathsf{funct}\ S), B \sqsubseteq \exists P_1\}$, and $\mathcal{A} = \{B(a)\}$.
The canonical interpretation $\mathcal{I}_\mathcal{K}$, depicted in Figure 1, clearly violates the functionality of P_1^- and hence is not a model of $\mathcal{K}$.

In general, $\mathcal{I}_\mathcal{K}$ cannot be a model of $\mathcal{K}$ since it is finite, and *DL-Lite*$^{\mathcal{N}}_{horn}$ does not enjoy the finite model property (FMP). A standard way to construct a (canonical) model from $\mathcal{I}_\mathcal{K}$ is to *unravel* it into a forest-shaped interpretation $\mathcal{U}_\mathcal{K}$ [2, 8]. We omit the definition of $\mathcal{U}_\mathcal{K}$ here and focus only in its properties:

Lemma 2. *Let $\mathcal{K}$ be a DL-Lite$^{\mathcal{N}}_{horn}$ knowledge base, and $\mathcal{U}_\mathcal{K}$ the unravelling of the canonical interpretation $\mathcal{I}_\mathcal{K}$, then the following hold:*

(p1) *$\mathcal{K}$ is satisfiable iff $\mathcal{U}_\mathcal{K} \models \mathcal{K}$.*
(p2) *For every DL-Lite$^{\mathcal{N}}_{horn}$ TBox axiom φ, $\mathcal{K} \models \varphi$ iff $\mathcal{U}_\mathcal{K} \models \varphi$.*

4 Finite Model Reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$

In this section, we study finite model reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$. Notably, the FMP it is already lost when considering only functionality constraints. Let us take the following *DL-Lite*$^{\mathcal{F}}_{horn}$ KB to illustrate this:

$$\mathcal{K}' = (\mathcal{T} \cup \{B \sqcap \exists P_2^- \sqsubseteq \bot\}, \mathcal{A}) \tag{1}$$

with $\mathcal{T}$ and $\mathcal{A}$ from Example 1. It is not hard to see that $\mathcal{K}'$ is satisfiable only by infinite models. Intuitively, in every model $\mathcal{I}$ of $\mathcal{K}'$, there is an infinite sequence of objects connected by P_1 and P_2 starting from $a^\mathcal{I}$: since a is an instance of B, $a^\mathcal{I}$ has a P_1-successor, d_1, and from $\exists P_1^- \sqsubseteq \exists P_2$, d_1 has a P_2 successor different from $a^\mathcal{I}$ (from $B \sqcap \exists P_2^- \sqsubseteq \bot$), say d_2, from $\exists P_2^- \sqsubseteq \exists P_1$, d_2 has a P_1-successor, d_3, different from d_1, (since P_1^- is functional), and d_3 has a P_2-successor, d_4, different from d_2 (since P_2^- is functional). These arguments can be used repeatedly to see that indeed an infinite number of objects are needed to satisfy the constraints in $\mathcal{K}'$.

In order to provide a method for reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$ w.r.t. finite models, we follow the approach taken by Cosmadakis et. al., [6] for characterizing finite implication of unary inclusion dependencies (UINDS) and functionality dependencies in databases. Given a *DL-Lite*$^{\mathcal{F}}_{horn}$-KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, we show that it is possible to 'enrich' $\mathcal{T}$ in such a way that it explicitly contains concept inclusions and functionality constraints that hold in every finite model of $\mathcal{T}$. We adapt the idea behind the axiomatization presented by Cosmadakis et. al., [6], and define a closure of a given TBox $\mathcal{T}$ in terms of arbitrary reasoning. Differently from what is done by Rosati [13], we do not exclude disjointness axioms of the form $B_1 \sqcap \ldots \sqcap B_k \sqsubseteq \bot$ from $\mathcal{T}$ for defining such a closure.

To simplify the presentation we consider an extension of *DL-Lite*$^{\mathcal{F}}_{horn}$ with axioms of the form $B_i \geq B_j$, with the following intended semantics for finite models: a finite interpretation $\mathcal{I}$ satisfies $B_i \geq B_j$ if and only if $\sharp(B_j)^\mathcal{I} \geq \sharp(B_j)^\mathcal{I}$.

Definition 2. *For a given DL-Lite$^{\mathcal{F}}_{horn}$ TBox $\mathcal{T}$,* $\mathsf{finClosure}(\mathcal{T})$ *denotes the minimal set of axioms satisfying the following conditions:*

1. $\mathcal{T} \subseteq \mathsf{finClosure}(\mathcal{T})$*;*
2. *For every pair of basic concepts B_1, B_2 occurring in $\mathcal{T}$. If* $\mathsf{finClosure}(\mathcal{T}) \models B_1 \sqsubseteq B_2$*, then* $B_2 \geq B_1 \in \mathsf{finClosure}(\mathcal{T})$*;*
3. *if* $(\mathsf{funct}\ R) \in \mathsf{finClosure}(\mathcal{T})$ *then* $\exists R \geq \exists R^- \in \mathsf{finClosure}(\mathcal{T})$*;*
4. *if* $\{B_1 \geq B_2, B_2 \geq B_3\} \subseteq \mathsf{finClosure}(\mathcal{T})$ *then* $B_1 \geq B_3 \in \mathsf{finClosure}(\mathcal{T})$*;*
5. *if* $\{(\mathsf{funct}\ R), \exists R^- \geq \exists R\} \subseteq \mathsf{finClosure}(\mathcal{T})$ *then* $(\mathsf{funct}\ R^-) \in \mathsf{finClosure}(\mathcal{T})$*;*
6. *if* $\mathsf{finClosure}(\mathcal{T}) \models B_1 \sqsubseteq B_2$*, and* $B_1 \geq B_2 \in \mathsf{finClosure}(\mathcal{T})$ *then* $B_2 \sqsubseteq B_1 \in \mathsf{finClosure}(\mathcal{T})$*.*

From 1, it follows that every model of $\mathsf{finClosure}(\mathcal{T})$ is also a model of $\mathcal{T}$. Since TBox reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$ is PTIME-complete [2], the following holds:

Proposition 1. $\mathsf{finClosure}(\mathcal{T})$ *can be computed in polynomial time on the size of $\mathcal{T}$.*

(1)-(4) in Definition 2 are based in logical consequences and are therefore sound w.r.t. arbitrary models. (5) and (6), on the other hand, are not sound w.r.t. infinite models, but a simple counting argument shows that they are sound w.r.t. finite models. Hence, we have the following result:

Lemma 3. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{F}}_{horn}$-TBox. Then, the following hold:*

(a) if $\mathsf{finClosure}(\mathcal{T}) \models \bigsqcap_k B_k \sqsubseteq B$ *then* $\mathcal{T} \models_{\mathsf{fin}} \bigsqcap_k B_k \sqsubseteq B$*;*
(b) if $(\mathsf{funct}\ R) \in \mathsf{finClosure}(\mathcal{T})$ *then* $\mathcal{T} \models_{\mathsf{fin}} (\mathsf{funct}\ R)$*.*

Moreover, the introduction of axioms of the form $B_i \geq B_j$, induces a directed graph $(\mathcal{V}, \mathcal{E})$, with $\mathcal{V}$ the set of concepts occurring in $\mathcal{T}$ and $(B_i, B_j) \in \mathcal{E}$ iff $B_i \geq B_j \in \mathsf{finClosure}(\mathcal{T})$. The implications w.r.t. finite models can be better understood by observing the structure of $(\mathcal{V}, \mathcal{E})$.

Example 2 (finClosure). Consider the TBox $\mathcal{T}$ from Example 1. $\mathsf{finClosure}(\mathcal{T})$ contains (among others) the axioms $\mathcal{T}_1 = \{\exists P_2 \sqsubseteq \exists P_1^-, \exists P_1 \sqsubseteq \exists P_2^-, (\mathsf{funct}\ P_1), (\mathsf{funct}\ P_2)\}$. Figure 2 shows a portion of the graph induced by $\mathsf{finClosure}(\mathcal{T})$. The dashed lines represent '$\geq$' inferred by concept inclusions, and the solid lines are '$\geq$' introduced by functionality assertions (rules 2–4). From a solid (dashed) edge (B_i, B_j) belonging to a cycle, it is inferred a solid (dashed) edge (B_j, B_i) (rules 5–6). In the example, from the edge $(d_{P_1}, d^-_{P_2})$, corresponding to the axiom $\exists P_2^- \sqsubseteq \exists P_1$, it is inferred that $\exists P_1 \sqsubseteq \exists P_2^- \in \mathsf{finClosure}(\mathcal{T})$. Analogously, from the solid line labelled with P_1^-, corresponding to $(\mathsf{funct}\ P_1^-)$ it is inferred that $(\mathsf{funct}\ P_1)$.
If there is an unsatisfiable concept B_i, this is reflected by an axiom of the form $\bot \geq B_i$. Let us consider $\mathcal{K}'$ from (1). We have that from the 'cycle rules', $\exists P_1 \sqsubseteq \exists P_2^- \in \mathsf{finClosure}(\mathcal{T}')$. Hence, $\mathsf{finClosure}(\mathcal{T}') \models \{\exists P_1 \sqsubseteq \exists P_2^-, B \sqsubseteq \exists P_1, B \sqcap \exists P_2^- \sqsubseteq \bot\}$, which implies that $\mathsf{finClosure}(\mathcal{T}') \models B \sqsubseteq \bot$, and then, by rule 1, $\bot \geq B \in \mathsf{finClosure}(\mathcal{T}')$ (see Figure 3). This means that in every finite model $\mathcal{I}$ of $\mathcal{T}'$, $\sharp B^{\mathcal{I}} = 0$. An inconsistency w.r.t. finite models is then derived from the ABox assertion $B(a)$.

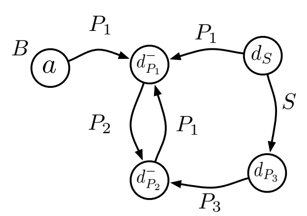

Fig. 1: $\mathcal{I}_\mathcal{K}$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$

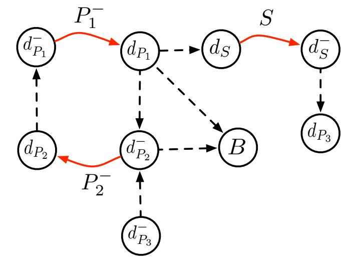

Fig. 2: $\mathsf{finClosure}(\mathcal{T})$

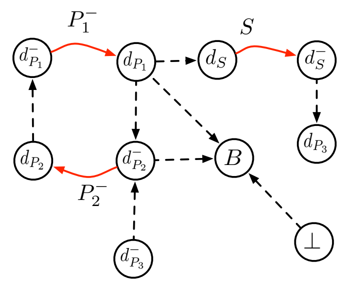

Fig. 3: $\mathsf{finClosure}(\mathcal{T}')$

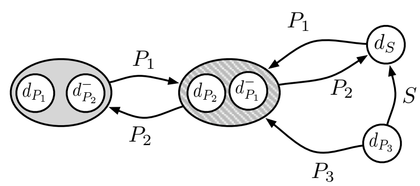

Fig. 4: $\mathcal{I}_{\widehat{\mathcal{T}}}$

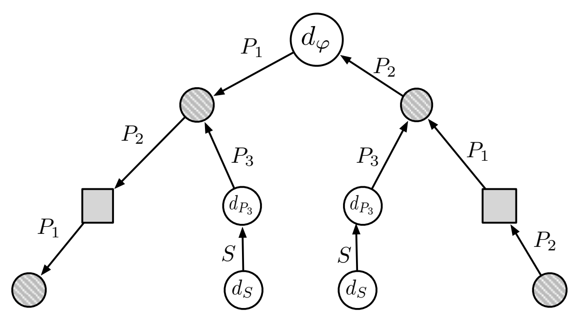

Fig. 5: $\mathcal{U}_{\widehat{\mathcal{T}}}$

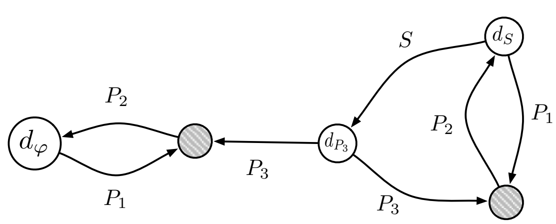

Fig. 6: $\mathcal{I}^f_\mathcal{T}$

We proceed to prove that finClosure is complete. More precisely, we show the following:

Lemma 4. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{F}}_{horn}$-TBox, and φ a DL-Lite$^{\mathcal{F}}_{horn}$ axiom on the signature of $\mathcal{T}$, i.e., φ is either a concept inclusion or a functionality constraint. If $\mathcal{T} \models_{\mathsf{fin}} \varphi$ then* $\mathsf{finClosure}(\mathcal{T}) \models \varphi$.

Proof. We shall show that $\mathsf{finClosure}(\mathcal{T}) \not\models \varphi$ implies $\mathcal{T} \not\models_{\mathsf{fin}} \varphi$. In what follows, we fix a *DL-Lite*$^{\mathcal{F}}_{horn}$-TBox $\mathcal{T}$, and set $\widehat{\mathcal{T}} = \mathsf{finClosure}(\mathcal{T})$. Then, by Lemma 2-**(p2)**, it suffices to show that $\mathcal{U}_{\widehat{\mathcal{T}}} \not\models \varphi$ implies $\mathcal{T} \not\models_{\mathsf{fin}} \varphi$, where $\mathcal{U}_{\widehat{\mathcal{T}}}$ is the unravelling of a canonical interpretation $\mathcal{I}_{\widehat{\mathcal{T}}}$ that depends on φ. Specifically, if $\varphi = C \sqsubseteq B$, then the 'root' of $\mathcal{U}_{\widehat{\mathcal{T}}}$ is an object $d \in C^{\mathcal{U}_{\widehat{\mathcal{T}}}} \setminus B^{\mathcal{U}_{\widehat{\mathcal{T}}}}$. On the other hand, if $\varphi = (\mathsf{funct}\ R)$, then $\mathcal{U}_{\widehat{\mathcal{T}}}$ is rooted at an object d with two different R-successors e and e'.

Let us consider the case $\varphi = C \sqsubseteq B$. We construct a finite model $\mathcal{I}^f_{\mathcal{T}}$ of $\mathcal{T}$ such that $\mathcal{U}_{\widehat{\mathcal{T}}} \not\models C \sqsubseteq B$ implies $\mathcal{I}^f_{\mathcal{T}} \not\models C \sqsubseteq B$. But first, we introduce some useful notation. For any two concepts B_1, B_2, we write $B_1 \sqsubseteq_{\widehat{\mathcal{T}}} B_2$ whenever $\widehat{\mathcal{T}} \models B_1 \sqsubseteq B_2$; and $B_1 \equiv_{\widehat{\mathcal{T}}} B_2$, if additionally $B_2 \sqsubseteq_{\widehat{\mathcal{T}}} B_1$. Since $\equiv_{\widehat{\mathcal{T}}}$ is an equivalence relation, the set of concepts $\mathfrak{E} = \{\exists R \mid R \in \mathit{role}^{\pm}(\widehat{\mathcal{T}})\}$ can be partitioned into equivalence classes w.r.t. $\equiv_{\widehat{\mathcal{T}}}$. Then, $[\exists R] \in \mathfrak{E}/\equiv_{\widehat{\mathcal{T}}}$ denotes the following equivalence class of concepts: $[\exists R] = \{B_i \in \mathfrak{E} \mid B_i \equiv_{\widehat{\mathcal{T}}} \exists R\}$. Before moving forward with the definition of $\mathcal{I}^f_{\mathcal{T}}$, we observe that the canonical interpretation $\mathcal{I}_{\widehat{\mathcal{T}}}$ constructed as in Definition 1 may introduce multiple witnesses for a given existential. We set $d_s = d'_s$ whenever $\exists S \equiv_{\widehat{\mathcal{T}}} \exists S'$. Therefore, domain of the canonical interpretation $\mathcal{I}_{\widehat{\mathcal{T}}}$ contains exactly one element d_S for each class $[\exists S] \in \mathfrak{E}/\equiv_{\widehat{\mathcal{T}}}$ (e.g., as in Figure 4).

We write $[\exists S_i] \xrightarrow{R} [\exists S_j]$ iff $\exists S_i \sqsubseteq_{\widehat{\mathcal{T}}} \exists R$ and $\exists R^- \in [\exists S_j]$. Analogously, $\exists S_i \geq_{\widehat{\mathcal{T}}} \exists S_j$ denotes that $\exists S_i \geq \exists S_j \in \widehat{\mathcal{T}}$.

We observe that $\geq_{\widehat{\mathcal{T}}}$ induces a coarser partition on $\mathfrak{E}$. For a concept $\exists S_i \in \mathfrak{E}$, the *cluster* of $\exists S_i$ is the set $\mathfrak{C}(\exists S_i) = \{\exists S_j \in \mathfrak{E} \mid \exists S_i \geq_{\widehat{\mathcal{T}}} \exists S_j \text{ and } \exists S_j \geq_{\widehat{\mathcal{T}}} \exists S_i\}$. In particular, for every two concepts $\exists S_i, \exists S_j \in \mathfrak{E}$, if $\exists S_i \equiv_{\widehat{\mathcal{T}}} \exists S_j$ then $\exists S_i \in \mathfrak{C}(S_j)$; but the implication on the other direction does not hold. Intuitively, if $\exists S_i, \exists S_j$ belong to the same cluster, then their extensions have the same cardinality in every finite model of $\widehat{\mathcal{T}}$, and of $\mathcal{T}$.

Further, we use $[\exists S_i] \succ [\exists S_j]$ to denote the fact that there exist concepts $\exists R \in [\exists S_i]$ and $\exists R' \in [\exists S_j]$ such that $\exists R \geq_{\widehat{\mathcal{T}}} \exists R'$, but $\exists R' \not\geq_{\widehat{\mathcal{T}}} \exists R$, i.e., $\exists R' \notin \mathfrak{C}(\exists R)$. For example, in the TBox $\mathcal{T}$ from Example 2, $[\exists P_1^-] \succ [\exists S]$. Notably, $\exists P_1^- \geq_{\widehat{\mathcal{T}}} \exists S$, but $\exists S \notin \mathfrak{C}(\exists P_1^-) = \{\exists P_1, \exists P_1^-, \exists P_2, \exists P_2^-\}$. It is also the case that $[\exists P_1^-] \not\succ [\exists P_2]$, since $\mathfrak{C}(\exists P_1^-) = \mathfrak{C}(\exists P_2)$.

For constructing the domain of the desired finite model $\mathcal{I}^f_{\mathcal{T}}$, we define the set of *finite paths* of $\mathcal{I}_{\widehat{\mathcal{T}}}$. $\sigma = (d_{S_0} \cdots d_{S_k}) \in \mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}})$ iff σ satisfies the following conditions:

1. $[\exists S_i] \xrightarrow{R} [\exists S']$, for some role R, such that :
   (a) $(\mathsf{funct}\ R^-) \in \widehat{\mathcal{T}}$,
   (b) $[\exists S_{i+1}] \in \mathfrak{C}(\exists S')$, and

(c) $\exists R \notin [\exists S_{i+1}]$

2. $[\exists S_{i+1}] \succ [\exists S_i]$

Intuitively, by condition (1a) a path $(\sigma \cdot d_R^-)$ can be 'reused' as a witness of an existential $\exists R$, whenever the inverse of R is not functional, otherwise a new object $(\sigma' \cdot d_R^-)$ is needed as a witness. Condition (1b) ensures that whenever such a witness path $(\sigma \cdot d_{S'})$ belongs to $\mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}})$, then also witnesses $(\sigma \cdot d_{R_i+1})$ for each class $[\exists R_{i+1}]$ in the cluster $\mathfrak{C}(\exists S')$ belong to $\mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}})$; condition (1c) avoids to include a witness that is already realized by $tail(\sigma)$. Moreover, by condition 2 the length of every path is bounded, and since $\mathfrak{C}$ is finite, then $\mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}})$ is a also finite.

We consider a subset of $\mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}})$ as the domain of $\mathcal{I}^f_{\mathcal{T}}$ that it is determined by $\varphi = C \sqsubseteq B$. More specifically, for $\sigma = d_S \cdot \sigma' \in \mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}})$, we write $\varphi \rightsquigarrow \sigma$ iff there is a sequence of roles $R_0, \ldots, R_n$ such that:

1. $C \sqcap \neg B \sqsubseteq \exists R_0$, $\exists S \in [\exists R_n^-]$;
2. for $i \leq n$, $\exists R_i \in [\exists S_i]$, $[\exists S_i] \xrightarrow{R_i} [\exists S_{i+1}]$, and $\exists R_i \notin [\exists S_{i+1}]$;
3. either $(\mathsf{funct}\ R_i^-) \notin \widehat{\mathcal{T}}$ or $[\exists S_{i+1}] \not\succ [\exists S_i]$.

We are ready now to define $\mathcal{I}^f_{\mathcal{T}}$. We set $\Delta^{\mathcal{I}^f_{\mathcal{T}}} = \{\sigma \in \mathsf{finpaths}(\mathcal{I}_{\widehat{\mathcal{T}}}) \mid \varphi \rightsquigarrow \sigma\}$. For each concept name A, $A^{\mathcal{I}^f_{\mathcal{T}}} \subseteq \Delta^{\mathcal{I}^f_{\mathcal{T}}}$, and for each atomic role P, $P \subseteq (\Delta^{\mathcal{I}^f_{\mathcal{T}}} \times \Delta^{\mathcal{I}^f_{\mathcal{T}}})$, such that:

$$
\begin{aligned}
A^{\mathcal{I}^f_{\mathcal{T}}} =& \{\sigma \in \Delta^{\mathcal{I}^f_{\mathcal{T}}} \mid tail(\sigma) \sqsubseteq_{\widehat{\mathcal{T}}} A\};\\
P_k^{\mathcal{I}^f_{\mathcal{T}}} =& \{((\sigma \cdot d_{S_i}), (\sigma \cdot d_{S_i} d_{S_j})) \mid [\exists S_i] \xrightarrow{P} [\exists S_j]\}\\
& \cup \{((\sigma \cdot d_{S_i} d_{S_j}), (\sigma \cdot d_{S_i})) \mid [\exists S_j] \xrightarrow{P^-} [\exists S_i]\}\\
& \cup \{(d_\varphi, (d_P^- \cdot \sigma)) \mid \varphi \sqsubseteq \exists P\}\\
& \cup \{((d_P^- \cdot \sigma), d_\varphi) \mid \varphi \sqsubseteq \exists P^-\}\\
& \cup \{((\sigma \cdot d_P), (\sigma', d_P^-)) \mid \sigma \neq \sigma', (\mathsf{funct}\ P^-) \notin \widehat{\mathcal{T}}\}.
\end{aligned}
$$

As an example, consider the model $\mathcal{I}^f_{\mathcal{T}}$, shown in Figure 6 for the TBox $\mathcal{T}$ from Example 2.

We claim that $\mathcal{I}^f_{\mathcal{T}}$ and $\mathcal{U}_{\widehat{\mathcal{T}}}$ are g-equisimilar. Indeed, a global g-equisimulation ρ can be defined by $(\sigma, \gamma) \in \rho$ iff $tail(\gamma) = d_R$, $tail(\sigma) = d_S$ and $\exists R \in [\exists S]$.

Since $\mathcal{U}_{\widehat{\mathcal{T}}} \models \widehat{\mathcal{T}}$, by Lemma 1*(b)*, $\mathcal{I}^f_{\mathcal{T}} \models \widehat{\mathcal{T}}$; and since $\mathcal{T} \subseteq \widehat{\mathcal{T}}$, $\mathcal{I}^f_{\mathcal{T}} \models \mathcal{T}$. Moreover, $\mathcal{I}^f_{\mathcal{T}}$ is as desired: $\mathcal{I}^f_{\mathcal{T}} \not\models C \sqsubseteq B$, since by construction $t_{\mathcal{I}^f_{\mathcal{T}}}(d_\varphi) = t_{\mathcal{U}_{\widehat{\mathcal{T}}}}(d_\varphi)$. Finally, the case for $\varphi = (\mathsf{funct}\ R)$, can be handled by a slight modification of the previous construction. Essentially, we substitute d_φ in the previous construction by a witness d_R with two R-successors.

From Lemma 3 and Lemma 4 we conclude that finite model TBox reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$ can be reduced to arbitrary TBox reasoning.

Theorem 1. *For a given DL-Lite*$^{\mathcal{F}}_{horn}$ *TBox* $\mathcal{T}$*, concepts* C_1 *and* C_2*. We have that the following hold:*

1. $\mathcal{T}$ *is finitely satisfiable iff* $\mathsf{finClosure}(\mathcal{T})$ *is satisfiable.*
2. $\mathcal{T} \models_{\mathsf{fin}} C_1 \sqsubseteq C_2$ *iff* $\mathsf{finClosure}(\mathcal{T}) \models C_1 \sqsubseteq C_2$.

Next, we show that the complexity of finite model reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$ remains in PTime, when considering also an ABox, i.e., the following hold:

Theorem 2. *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ *be a DL-Lite*$^{\mathcal{F}}_{horn}$ *KB. Then* $\mathcal{K}$ *is finitely satisfiable iff* $(\mathsf{finClosure}(\mathcal{T}), \mathcal{A})$ *is satisfiable.*

Thus, the complexity of reasoning in *DL-Lite*$^{\mathcal{F}}_{horn}$ coincides for finite and arbitrary models.

Theorem 3. *Finite model reasoning in DL-Lite*$^{\mathcal{F}}_{horn}$ *is* PTime*-complete.*

As pointed out in Section 3 the canonical interpretation of a knowledge base $\mathcal{K}$ constructed as in Definition 1 it is not in general a model of $\mathcal{K}$, due to the presence of functionality constraints (and arbitrary number restrictions in general). The latter observation provides an intuition for the construction of $\mathcal{I}^{f}_{\mathcal{T}}$. Intuitively, $\mathcal{I}_{\widehat{\mathcal{T}}}$ can be transformed into a finite model by creating 'copies' of certain portions (clusters) in order to resolve violations to functionality constraints; then, although the number of R-successors of some objects in the model increases (specifically for those roles R in the TBox such that $(\mathsf{funct}\ R) \notin \widehat{\mathcal{T}}$), this does not trigger any inconsistency, because the expressive power of *DL-Lite*$^{\mathcal{F}}_{horn}$ allows only to distinguish between two types of objects: those with exactly one R-successor, and those with one or more. As we shall see on the next section this approach for constructing a finite model fails when considering arbitrary number restrictions.

5 Finite Model Reasoning in *DL-Lite*$^{\mathcal{N}}_{horn}$

Kontchakow et. al., [9] show the following result by a reduction of the SAT problem to finite satisfiability in *DL-Lite*$^{\mathcal{N}}_{horn}$.

Lemma 5 ([9, Remark 98]). *Finite satisfiability of DL-Lite*$^{\mathcal{N}}_{horn}$ *TBoxes is* NP*-hard.*

From the proof of the previous lemma, it can be seen that, contrary to the arbitrary model case, when restricting to finite models in *DL-Lite*$^{\mathcal{N}}_{horn}$, it is possible to express disjunctive knowledge, such as covering of a concept C by a disjunction of concepts, even though the disjunction operator, '$\sqcup$', is not part of the logic. Moreover, *DL-Lite*$^{\mathcal{N}}_{horn}$ looses convexity when restricting to finite models. In order to understand this more clearly, let us consider a TBox $\mathcal{T}$ with the following axioms:

$$\geq 3\ P_1 \sqsubseteq \bot, \quad \exists P_1 \sqsubseteq \geq 2\ P_1, \quad \geq 2\ P_1 \equiv \exists P_2, \quad \top \sqsubseteq \exists P_1^-, \qquad (\mathsf{funct}\ P_1^-),$$
$$B_2 \equiv \exists P_2^-, \quad (\mathsf{funct}\ P_2), \qquad (\mathsf{funct}\ P_2^-), \qquad B_1 \sqcap B_2 \sqsubseteq \bot.$$

Let $\mathcal{I}$ be a finite model of $\mathcal{T}$, and $N = \sharp(\Delta^{\mathcal{I}})$. We have that $N = 2 \cdot \sharp(\geq 2\ P_1)^{\mathcal{I}}$. Furthermore, $\sharp(\geq 2\ P_1) = \sharp(B_2)^{\mathcal{I}} = M$, since $P_2^{\mathcal{I}}$ is a bijective function; and since B_1 is disjoint with B_2, then $\sharp(B_1)^{\mathcal{I}} = M$. Hence, $\Delta^{\mathcal{I}} = (B_1)^{\mathcal{I}} \cup (B_2)^{\mathcal{I}}$; and as a consequence $\mathcal{T} \models_{\mathsf{fin}} \geq 2\ P_1 \sqsubseteq B_1 \sqcup B_2$. However, $\mathcal{T} \not\models \geq 2\ P_1 \sqsubseteq B_1$, and $\mathcal{T} \not\models \geq 2\ P_1 \sqsubseteq B_2$.

The best known upper bound for finite satisfiability in *DL-Lite*$^{\mathcal{N}}_{horn}$ is ExpTime, which is given by the complexity of finite model reasoning in $\mathcal{ALCQI}$ [11]. The approach taken by Lutz et. al., [11] is to transform a given $\mathcal{ALCQI}$ TBox into a system of linear inequalities which is exponential on the size of the TBox. We conjecture that this exponential blow up can be avoided when considering *DL-Lite*$^{\mathcal{N}}_{horn}$ TBoxes. The combinatorial nature of this problem suggests indeed the use of techniques of linear programming. However, we consider that a reduction of this problem to the fragment of FOL with one variable and counting quantifiers is also feasible. For devising ad hoc algorithms for finite model reasoning in DLs, it is still relevant to propose a constructive approach as in the case of *DL-Lite*$^{\mathcal{F}}_{horn}$. All these research problems, as well as constructions of finite models of KBs in logics in the $\mathcal{EL}$ family constitute ongoing research.

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