Attribute Exploration in a Fuzzy Setting

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Abstract. Since its development attribute exploration was successfully applied in different fields, proving itself as a strong tool for knowledge acquisition. However, the disadvantage of this method is that it can be applied only for binary data. The growing number of applications of fuzzy logic in numerous domains including formal concept analysis makes it a natural wish to generalise the powerful technique of attribute exploration for fuzzy data. It is this paper's purpose to fulfill this wish and present a generalisation of attribute exploration to the fuzzy setting.

Keywords: Attribute exploration, knowledge discovery, fuzzy data

1 Introduction

Attribute exploration, as introduced in [1], is a tool for knowledge discovery by interactive determination of the implications holding between a given set of attributes. This method is especially useful when the examples, objects having the considered attributes, are infinite, hardly to enumerate or (partially) unknown. The user is asked whether some implications (the smallest set of implications from which all the other implications can be derived) hold. If the answer is affirmative, the next implication is considered. If, however, the implication is false, the user has to provide a counterexample. This method assumes that the user can distinguish between true and false implications and that he can provide counterexamples for false implications. The result of the attribute exploration is a set of implications which are true in general for the attributes under consideration and a representative set of examples for the whole theory.

Attribute exploration was successfully applied in different areas of research, for a brief overview see Subsection 2.1.

Formal fuzzy concept analysis goes back to [2, 3]. Its need arose by the fact that objects can have attributes with some truth degree instead of either having or not having them, reflecting that life is not just black and white. In such a fuzzy setting one can also be interested in the implications between attributes. These are formulas like $A \Rightarrow B$, where A and B are fuzzy sets of attributes. Such implications can be interpreted in fuzzy contexts, meaning that *if objects have the attributes from A to at least the degree a, then they also have the attributes from B to at least the degree b.* Attribute implications in a fuzzy setting were mainly developed and investigated by R. Belohlavek and V. Vychodil in a series of papers, see for example [4, 5]. Due to the large number of fuzzy attribute implications in a formal fuzzy context, one is interested in the smallest set of attribute implications, the so-called *stem base*, from which all the other implications can be derived. The problem of determining the stem bases for the crisp case was studied in [6], see also [1]. However, in the fuzzy setting these stem bases need neither to be unique nor to exist. These facts split the problem of fuzzy attribute exploration into two cases, as we will see in Sections 3 and 4. We will show under which conditions an attribute exploration in a fuzzy setting can be performed successfully. The research in attribute exploration in the fuzzy setting is still at its beginning. We expect for it at least the same popularity in applications as its crisp variant has gained.

The article is structured as follows: In Section 2 we give short introductions to attribute exploration in the crisp setting, fuzzy sets and fuzzy logic, formal fuzzy concept analysis and implications in such a setting. Section 3 first presents how the stem bases can be computed in a fuzzy setting using the globalisation and afterwards it focuses on attribute exploration in such a setting. In Section 4 we treat the same subject as in the section before but this time we use a general hedge in the residuated lattice for the exploration. The last section contains concluding remarks and further topics of research.

2 Preliminaries

2.1 Crisp Attribute Exploration

We assume basic familiarities with Formal Concept Analysis and refer the reader to [1].

Attribute exploration ([1]) permits the interactive determination of the implications holding between the attributes of a given context. However, there are situations when the object set of a context is too large (possibly infinite) or difficult to enumerate. With the examples (possibly none) of our knowledge we build the object set of the context step-by-step. The stem base of this context is built stepwise and we are asked whether the implications of the base are true. If an implication holds, then it is added to the stem base. If however, an implication does not hold, we have to provide a counterexample. While performing an attribute exploration we have to be able to distinguish between true and false implications and to provide correct counterexamples for false implications. This is a crucial point since the algorithm is naive and will believe whatever we tell it. Once a decision was taken about the validity of an implication the choice cannot be reversed. Therefore, the counterexamples may not contradict the so-far confirmed implications. The procedure ends when all implications of the current stem base hold in general. This way we obtain an object set which is representative for the entire theory, theory which may also be infinite.

The following proposition justifies why we do not have to reconsider the already confirmed implications:

Proposition 1. *([1]) Let* $\mathbb{K}$ *be a context and* $P_1, P_2, \ldots, P_n$ *be the first* n *pseudo-intents of* $\mathbb{K}$ *with respect to the lectic order. If* $\mathbb{K}$ *is extended by an object* g *the* *object intent $g^{\uparrow}$ of which respects the implications $P_i \to P_i^{\downarrow\uparrow}$, $i \in \{1, \ldots, n\}$, then $P_1, P_2, \ldots, P_n$ are also the lectically first n pseudo-intents of the extended context.*

As mentioned in the introductory section, attribute exploration was successfully applied in both theoretical and practical research domains. On the one hand it facilitated the discovery of implications between properties of mathematical structures, see for example [7–9]. On the other hand it was also used in real-life scenarios, for instance in civil engineering ([10]), chemistry ([11]), information systems ([12]), etc.

The algorithm is implemented in different formal concept analytical tools, as for example in ConExp[1] and Conexp-clj[2].

There are also further variants of attribute exploration, for instance *attribute exploration with background knowledge* for the case that the user knows in advance some implications between the attributes that hold ([13, 14]). Another possibility is to perform *concept exploration* as presented in [15]. By replacing the implications with Horn clauses from predicate logic one obtains the so-called *rule exploration* developed in [16].

2.2 Fuzzy Sets and Fuzzy Logic

In this subsection we present some basics about fuzzy sets and fuzzy logic. The interested reader may find more details for instance in [17, 3].

A **complete residuated lattice with truth-stressing hedge** (shortly, a hedge) is an algebra $\mathbf{L} := (L, \wedge, \vee, \otimes, \to, ^*, 0, 1)$ such that: $(L, \wedge, \vee, 0, 1)$ is a complete lattice; $(L, \otimes, 1)$ is a commutative monoid; 0 is the least and 1 the greatest element; the adjointness property, i.e., $a \otimes b \leq c \Leftrightarrow a \leq b \to c$, holds for all $a, b, c \in L$. The hedge * is a unary operation on L satisfying the following:

i) $a^* \leq a$,
ii) $(a \to b)^* \leq a^* \to b^*$,
iii) $a^{**} = a^*$,
iv) $\bigwedge_{i \in I} a_i^* = (\bigwedge_{i \in I} a_i)^*$,

for every $a, b, a_i \in L$ $(i \in I)$. Elements of L are called **truth degrees**, $\otimes$ and $\to$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". The hedge * is a (truth function of) logical connective "very true", see [17, 18]. Properties (i)-(iv) have natural interpretations, i.e., (i) can be read as "if a is very true, then a is true", (ii) can be read as "if $a \to b$ is very true and if a is very true, then b is very true", etc. From the mathematical point of view, the hedge operator is a special kernel operator which controls the size of the fuzzy concept lattice.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$, $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\to$. The

[1] http://conexp.sourceforge.net/
[2] http://daniel.kxpq.de/math/conexp-clj/

three most important pairs of adjoint operations on the unit interval are:

Lukasiewicz: $a \otimes b := max(0, a + b - 1)$ with $a \to b := min(1,\ 1 - a + b)$,

Gödel: $a \otimes b := min(a, b)$ with $a \to b := \begin{cases} 1, \ a \leq b \\ b, \ a \gneqq b \end{cases}$,

Product: $a \otimes b := ab$ with $a \to b := \begin{cases} 1, \quad a \leq b \\ b/a, \ a \gneqq b \end{cases}$.

Typical examples for the hedge are the *identity*, i.e., $a^* := a$ for all $a \in L$, and the *globalization*, i.e., $a^* := 0$ for all $a \in L \setminus \{1\}$ and $a^* := 1$ if and only if $a = 1$.

Let $\mathbf{L}$ be the structure of truth degrees. A **fuzzy set** (**L-set**) A in a universe U is a mapping $A : U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$, then A can be denoted by $A = \{^{a_1}/u_1, \ldots, ^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i \in \{1, \ldots, n\}$. Let $\mathbf{L}^U$ denote the collection of all fuzzy sets in U. The operations with fuzzy sets are defined component-wise. For example, the intersection of fuzzy sets $A, B \in \mathbf{L}^U$ is a fuzzy set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary fuzzy relations (**L**-relations) between G and M can be thought of as fuzzy sets in the universe $G \times M$. For $A, B \in \mathbf{L}^U$, the **subsethood degree** is defined as

$$S(A, B) := \bigwedge_{u \in U} (A(u) \to B(u)),$$

which generalises the classical subsethood relation $\subseteq$. Therefore, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$.

2.3 Formal Fuzzy Concepts and Concept Lattices

In the following we give brief introductions to Formal Fuzzy Concept Analysis [2, 3].

A triple (G, M, I) is called a **formal fuzzy context** if $I : G \times M \to L$ is a fuzzy relation between the sets G and M and L is the support set of some residuated lattice. Elements from G and M are called **objects** and **attributes**, respectively. The fuzzy relation I assigns to each $g \in G$ and each $m \in M$ the truth degree $I(g, m) \in L$ to which the object g has the attribute m. For fuzzy sets $A \in \mathbf{L}^G$ and $B \in \mathbf{L}^M$ the **derivation operators** are defined by

$$A^{\uparrow}(m) := \bigwedge_{g \in G} (A(g)^* \to I(g, m)),\ B^{\downarrow}(g) := \bigwedge_{m \in M} (B(m) \to I(g, m)), \quad (1)$$

for $g \in G$ and $m \in M$. Then, $A^{\uparrow}(m)$ is the truth degree of the statement "m is shared by all objects from A" and $B^{\downarrow}(g)$ is the truth degree of "g has all attributes from B". The operators $^{\uparrow}, ^{\downarrow}$ form a so-called Galois connection with hedges ([19]). A **formal fuzzy concept** is a tuple $(A, B) \in \mathbf{L}^G \times \mathbf{L}^M$ such that

$A^{\uparrow} = B$ and $B^{\downarrow} = A$. Then, A is called the **(fuzzy) extent** and B the **(fuzzy) intent** of (A, B). We denote the set of all fuzzy concepts of a given context (G, M, I) by $\mathfrak{B}(G^*, M, I)$. Concepts serve for classification. Consequently, the super- and subconcept relation plays an important role. A concept is called superconcept of another if it is more general, i.e., if it contains more objects. More formally, (A_1, B_1) is a **subconcept** of (A_2, B_2), written $(A_1, B_1) \leq (A_2, B_2)$, iff $A_1 \subseteq A_2$ (iff $B_1 \supseteq B_2$). Then, we call (A_2, B_2) the **superconcept** of (A_1, B_1). The set of all fuzzy concepts ordered by this concept order forms a complete fuzzy lattice (with hedge), the so-called **fuzzy concept lattice** which is denoted by $\underline{\mathfrak{B}}(G^*, M, I) := (\mathfrak{B}(G^*, M, I), \leq)$, see [20].

The *fuzzy lectic order* ([21]) is defined as follows: Let $L = \{l_0 < l_1 < \cdots < l_n\}$ be the support set of some residuated lattice. For $a := (i, j)$ and $b := (h, k)$, where $a, b \in M \times L$, we write

$$a \leq b :\Longleftrightarrow (i < h) \text{ or } (i = h \text{ and } l_j \geq l_k).$$

For $B \in \mathbf{L}^M$ and $(i, j) \in M \times L$ we define

$$B \oplus (i, j) := ((B \cap \{1, 2, \ldots, i-1\}) \cup \{{}^{a_j}/i\})^{\downarrow\uparrow}.$$

Furthermore, for $B, C \in \mathbf{L}^M$ define

$$B <_{(i,j)} C :\Longleftrightarrow B \cap \{1, \ldots, i-1\} = C \cap \{1, \ldots, i-1\} \text{ and } B(i) < C(i) = a_j.$$

We say that B is **lectically smaller** than C, written $B < C$, if $B <_{(i,j)} C$ for some (i, j). As in the crisp case we have that $B^+ := B \oplus (i, j)$ is the least intent which is greater than a given B with respect to $<$ and (i, j) is the greatest with $B <_{(i,j)} B \oplus (i, j)$.

Example 1. Consider the formal fuzzy context (G, M, I) given in Figure 1. Using the Lukasiewicz logic with the identity as hedge we obtain 15 formal fuzzy concepts. For example $(\{Mo, T, {}^{0.5}/W\}, \{c, r\})$ is a fuzzy concept. We could name it the concept of *cold and rainy days* because of its intent. Then, Monday, Tuesday and partially Wednesday belong to this concept, i.e., they are cold and rainy days. Another example is $(\{{}^{0.5}/W, Th, F\}, \{w\})$ which corresponds to *warm days*. Yet another example are the *warm and partially rainy days* given by $(\{{}^{0.5}/W, Th, {}^{0.5}/F\}, \{w, {}^{0.5}/r\})$. The fuzzy concept lattice is displayed on the left side in Figure 2. For better legibility we did not use all the labels. Using the globalisation instead of the identity, we obtain 10 formal fuzzy concepts which are displayed on the right in Figure 2. The concepts obtained through the globalisation need not be a subset of those obtained with the identity. In this example this case does not appear. Using the Gödel structure one obtains 13 concepts with the identity and 10 with the globalisation.

2.4 Fuzzy Implications and Non-redundant Bases

As already mentioned, fuzzy implications were studied in a series of papers by R. Belohlavek and V. Vychodil, for instance in [4, 5].

	warm (w)	cold (c)	rainy (r)
Monday (Mo)	0	1	1
Tuesday (T)	0	1	1
Wednesday (W)	0.5	0.5	1
Thursday (Th)	1	0	0.5
Friday (F)	1	0	0

Fig. 1. Example of a fuzzy formal context

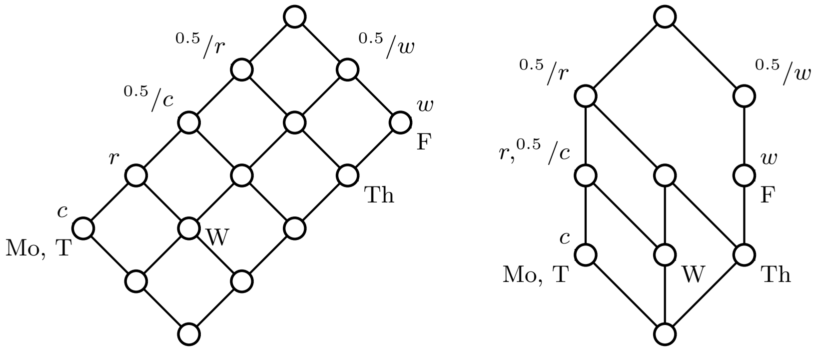

Fig. 2. Formal fuzzy concept lattices

A **fuzzy attribute implication** (over the attribute set M) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^M$. The verbal meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it also has all attributes from B". The notions "being very true", "to have an attribute", and logical connective "if-then" are determined by the chosen $\mathbf{L}$. For a fuzzy set $N \in \mathbf{L}^M$ of attributes, the degree $||A \Rightarrow B||_N \in L$ to which $A \Rightarrow B$ is valid in N is defined as

$$||A \Rightarrow B||_N := S(A, N)^* \to S(B, N).$$

If N is the fuzzy set of all attributes of an object g, then $||A \Rightarrow B||_N$ is the truth degree to which $A \Rightarrow B$ holds for g. For a set $\mathcal{N} \subseteq \mathbf{L}^M$, the degree $||A \Rightarrow B||_{\mathcal{N}} \in L$ to which the implication $A \Rightarrow B$ holds in $\mathcal{N}$ is defined by

$$||A \Rightarrow B||_{\mathcal{N}} := \bigwedge_{N \in \mathcal{N}} ||A \Rightarrow B||_N.$$

For a fuzzy context (G, M, I), let $I_g \in \mathbf{L}^M$ $(g \in G)$ be a fuzzy set of attributes such that $I_g(m) = I(g, m)$ for each $m \in M$. Clearly, I_g corresponds to the row labelled g in (G, M, I). The degree $||A \Rightarrow B||_{(G,M,I)} \in L$ to which $A \Rightarrow B$ holds in (each row of) $\mathbb{K} = (G, M, I)$ is defined by

$$||A \Rightarrow B||_{\mathbb{K}} = ||A \Rightarrow B||_{(G,M,I)} := ||A \Rightarrow B||_{\mathcal{N}},$$

where $\mathcal{N} := \{I_g \mid g \in G\}$. Denote by

$$\text{Int}(G^*, M, I) := \{B \in \mathbf{L}^M \mid (A, B) \in \mathfrak{B}(G^*, M, I) \text{ for some } A\}$$

the set of all intents of $\mathfrak{B}(G^*, M, I)$. Since $N \in \mathbf{L}^M$ is the intent of some concept if and only if $N = N^{\downarrow\uparrow}$, we have $\text{Int}(G^*, M, I) = \{N \in \mathbf{L}^M \mid N = N^{\downarrow\uparrow}\}$. The degree $||A \Rightarrow B||_{\mathfrak{B}(G^*,M,I)} \in L$ to which $A \Rightarrow B$ holds in (the intents of) $\mathfrak{B}(G^*, M, I)$ is defined by

$$||A \Rightarrow B||_{\mathfrak{B}(G^*,M,I)} := ||A \Rightarrow B||_{\text{Int}(G^*,M,I)}.$$

Lemma 1. *([22]) Let (G, M, I) be a fuzzy context. Then,*

$$||A \Rightarrow B||_{(G,M,I)} = ||A \Rightarrow B||_{\mathfrak{B}(G^*,M,I)} = S(B, A^{\downarrow\uparrow})$$

for each fuzzy attribute implication $A \Rightarrow B$.

Example 2. Consider once again the fuzzy context given in Figure 1. Using the Lukasiewicz logic and the globalisation as the hedge we have $||c \Rightarrow r||_{(G,M,I)} = 1$, i.e., this is a true implication. However, in the fuzzy case, there are implications which are valid to a certain degree different from 1, for instance we have the implication $||c \Rightarrow \{^{0.5}/w, r\}||_{(G,M,I)} = 0.5$. We obtain the same truth value for these implications also by using the identity. Consider the Gödel logic with the globalisation. For example, we have the implication $||w, r \Rightarrow c||_{(G,M,I)} = 1$ but using the identity this implication holds with the truth value 0. This is due to the fact that we have $\{w, r\}^{\downarrow\uparrow} = \{w, r, c\}$ with the globalisation and $\{w, r\}^{\downarrow\uparrow} = \{w, r\}$ with the identity.

Due to the large number of implications in a fuzzy and even in a crisp formal context, one is interested in the stem base of the implications. The **stem base** is a set of implications which is non-redundant and complete. The problem for the fuzzy case was studied in [5, 22, 23]. Neither the existence nor the uniqueness of the stem base for a given fuzzy context is guaranteed in general. How these problems can be overcome is the topic of the rest of this subsection. For a more detailed description we refer the reader to the papers cited above.

Let T be a set of fuzzy attribute implications. A fuzzy attribute set $N \in \mathbf{L}^M$ is called a **model** of T if $||A \Rightarrow B||_N = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $Mod\,(T)$, i.e.,

$$Mod\,(T) := \{N \in \mathbf{L}^M \mid N \text{ is a model of } T\}.$$

The degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ **semantically follows** from T is defined by $||A \Rightarrow B||_T := ||A \Rightarrow B||_{Mod(T)}$. T is called **complete** (in (G, M, I)) if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{(G,M,I)}$ for each $A \Rightarrow B$. If T is complete and no proper subset of T is complete, then T is called a **non-redundant basis**.

Theorem 1. *([5]) T is complete iff $Mod\,(T) = Int(G^*, M, I)$.*

As in the crisp case the stem base of a given fuzzy context can be obtained through the pseudo-intents.

Definition 1. *$\mathcal{P} \subseteq \mathbf{L}^M$ is called a **system of pseudo-intents** if for each $P \in \mathbf{L}^M$ we have:*

$$P \in \mathcal{P} \iff (P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P).$$

For each (G, M, I) there exists a unique system of pseudo-intents, if * is the globalisation and M is finite (this does not hold for the other hedges in general).

Theorem 2. *([22]) $T := \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ is complete and non-redundant. If * is the globalization, then T is unique and minimal.*

3 Fuzzy Attribute Exploration with Globalisation

Attribute exploration is a very powerful tool. However, its theoretical basis lies in Proposition 1 which represents its key to success. Thus, the crucial step is to generalise this proposition to the fuzzy setting. After developing the theoretical ingredients for a successful attribute exploration in a fuzzy setting, we turn our attention to its practical parts. First, we develop an appropriate algorithm for this technique and afterwards illustrate the method by an example.

In case we choose for * the globalisation, then the formalisation of pseudo-intents from Definition 1 becomes: $\mathcal{P} \subseteq \mathbf{L}^M$ is a system of pseudo-intents if

$$P \in \mathcal{P} \iff (P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for each } Q \in \mathcal{P} \text{ with } Q \subsetneqq P). \qquad (2)$$

Theorem 3. *([22]) Let $\mathbf{L}$ be a residuated lattice with globalization. Then, for each (G, M, I) with finite M there is a unique system of pseudo-intents $\mathcal{P}$ given by* (2).

For $Z \in \mathbf{L}^M$ we put

$$\begin{aligned} Z^{T^*} &:= Z \cup \bigcup \{B \otimes S(A, Z)^* \mid A \Rightarrow B \in T \text{ and } A \neq Z\}, \\ Z^{T^*_0} &:= Z, \\ Z^{T^*_n} &:= (Z^{T^*_{n-1}})^{T^*}, \text{ for } n \geq 1, \end{aligned}$$

where $B \otimes S(A, Z)^*$ is computed component-wise, and we define an operator cl_{T^*} on $\mathbf{L}$-sets in M by

$$cl_{T^*}(Z) := \bigcup_{n=0}^{\infty} Z^{T^*_n}.$$

Theorem 4. *([5]) If * is the globalisation, then cl_{T^*} is an $\mathbf{L}^*$-closure operator and*

$$\{cl_{T^*}(Z) \mid Z \in \mathbf{L}^M\} = \mathcal{P} \cup \mathit{Int}(X^*, Y, I).$$

According to this theorem, if * is the globalisation, then we can obtain all intents and all pseudo-intents of a given fuzzy context by computing the fixed points of cl_{T^*}. In [5] an algorithm for the computation of all intents and all pseudo-intents in lectic order was proposed. Therefore, the following result holds:

Proposition 2. *Let* $\mathbf{L}$ *be a residuated lattice with hedge and let* * *be the globalisation. Further, let* $\mathcal{P}$ *be the unique system of pseudo-intents of the fuzzy context* (G, M, I) *such that* $P_1, P_2, \ldots, P_n \in \mathcal{P}$ *are the first* n *pseudo-intents in* $\mathcal{P}$ *with respect to the lectic order. If* (G, M, I) *is extended by an object* g *the object intent* $g^{\uparrow}$ *of which respects the implications* $P_i \to P_i^{\downarrow\uparrow}$, $i \in \{1, \ldots, n\}$, *then* $P_1, P_2, \ldots, P_n$ *remain the lectically first* n *pseudo-intents of the extended context.*

Proof. Easy, by induction on the number of pseudo-intents in $\mathcal{P}$.

With this result we are able to generalise the attribute exploration algorithm to the fuzzy setting, as displayed below.

(1) $\mathcal{L} := \varnothing$; $A := \varnothing$
(2) **if** $(A = A^{\downarrow\uparrow})$
(3)   **then add** A to $\mathrm{Int}(\mathbb{K})$
(4)   **else** Ask expert whether $||A \Rightarrow A^{\downarrow\uparrow}||_{\mathbb{K}} = 1$
(5)   If yes, **add** $A \Rightarrow A^{\downarrow\uparrow}$ to $\mathcal{L}$
(6)   **else** ask for counterexample g and **add** it to $\mathbb{K}$
(7) **end if**
(8) **do while** $(A \neq M)$
(9)   **for** $i = n, \ldots, 1$ **and for** $l = \max L, \ldots, \min L$ **with** $A(i) < l$ **do**
(10) $B := cl_{T^*}(A)$
(11)   **if** $(A \searrow i = B \searrow i)$ **and** $(A(i) < B(i))$ **then**
(12) $A := B$
(13)   **if** $(A = A^{\downarrow\uparrow})$
(14)   **then add** A to $\mathrm{Int}(\mathbb{K})$
(15)   **else** Ask expert whether $||A \Rightarrow A^{\downarrow\uparrow}||_{\mathbb{K}} = 1$
(16)   If yes, **add** $A \Rightarrow A^{\downarrow\uparrow}$ to $\mathcal{L}$
(17)   **else** ask for counterexample g and **add** it to $\mathbb{K}$
(18)   **end if**
(19)   **end if**
(20)   **end for**
(21) **end do**

Fig. 3. Algorithm for attribute exploration with globalisation

The first intent or pseudo intent is the empty set. If it is an intent, add it to the set of intents of the context. Otherwise, ask the expert whether the implication is true in general. If so, add this implication to the stem base else ask for a counterexample and add it to the context (line $2-6$). Until A is different from the whole attribute set, repeat the following steps: Search for the largest attribute i in M with its largest value l such that $A(i) < l$. For this attribute compute its closure with respect to the cl_{T^*}-closure operator and check whether the result is the lectically next intent or pseudo-intent (line $9-12$). Thereby,

$A \searrow i := A \cap \{1, \ldots, i-1\}$. If the result is an intent, add it to the set of intents (line $13-14$), otherwise ask the user whether the implication provided by the pseudo-intent holds. If the implication holds, add it to the stem base otherwise ask the user for a counterexample (line $15-17$).

The algorithm generates interactively the stem base of the formal fuzzy context. As in the crisp case we enumerate the intents and pseudo-intents in the lectic order. Hence, we go through the list of all such elements. Due to Proposition 2 we are allowed to extend the context by objects whose object intents respect the already confirmed implications. This way, the pseudo-intents already used in the stem base do not change. Hence, the algorithm is sound and correct.

Example 3. We want to explore the size and distance of the planets. We include some of them into the object set and obtain the context given in Figure 4. In this example we will be using the Lukasiewicz logic with the globalisation as hedge.

	small (s)	large (l)	far (f)	near (n)
Earth	1	0	0	1
Mars	1	0	0.5	1
Pluto	1	0	1	0

Fig. 4. Initial context

We start the attribute exploration. The first pseudo-intent is $\varnothing$ and we are asked

All objects have the attribute s to degree 1?

This is of course not true and we provide a counterexample:

	small (s)	large (l)	far (f)	near (n)
Jupiter	0	1	1	0.5

The next pseudo-intent is n and we are asked

Objects having attribute n to degree 1 also have attribute s to degree 1?

This is a true implication and we confirm it. The next pseudo-intent is $\{f, {}^{0.5}/n\}$ which yields the following question:

Objects having attribute f and n to degree 1 and 0.5, respectively, also have attribute l to degree 1?

This is a true implication and we confirm it. The algorithm proceeds with

Objects having attribute l to degree 0.5 also have the attributes l, f, n to degree $1, 1, 0.5$, respectively?

This implication is not true for our planet system and we give a counterexample:

	small (s)	large (l)	far (f)	near (n)
Uranus	0.5	0.5	1	0

The following four implications are true, so we will confirm them:

$$\begin{aligned}&{}^{0.5}/l \Rightarrow f,\\ &l, f \Rightarrow^{0.5} /n,\\ &{}^{0.5}/s,^{0.5} /n \Rightarrow s, n,\\ &s,^{0.5} /l, f \Rightarrow l, n.\end{aligned}$$

And the attribute exploration has stopped. Now we have an extended formal fuzzy context, namely the one containing Jupiter and Uranus besides the objects given in Figure 4. Note that we did not have to include all the planets into the object set, just a representative part of them. The other planets with their attributes are displayed in Figure 5. These objects contain just redundant information and the knowledge provided by them is already incorporated into the stem base of the extended context.

	small (s)	large (l)	far (f)	near (n)
Mercury	1	0	0	1
Venus	1	0	0	1
Saturn	0	1	1	0.5
Neptune	0.5	0.5	1	0

Fig. 5. Superfluous planets

4 Fuzzy Attribute Exploration with General Hedges

As the title of this section suggests, we will now turn our attention to attribute exploration with general hedges. After introducing the necessary background information, we will focus on the exploration. As it turns out, there are several obstacles that make a straight-forward generalisation of attribute exploration in such a setting impossible. At the end of the section we will discuss which approaches may lead to a successful exploration. However, it is also an open question whether an exploration in such a setting is desirable.

The computation of the systems of pseudo-intents for general hedges was studied in [23]. For a fuzzy context (G, M, I) we compute the following:

$$V := \{P \in \mathbf{L}^M \mid P \neq P^{\downarrow\uparrow}\}, \tag{3}$$

$$E := \{(P, Q) \in V \times V \mid P \neq Q \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P \neq 1\}. \tag{4}$$

In case of a non-empty V, $\mathbf{G} := (V, E \cup E^{-1})$ is a graph. For $Q \in V$, $\mathcal{P} \subseteq V$ define the following subsets of V:

$$Pred\,(Q) := \{P \in V \mid (P, Q) \in E\},$$
$$Pred\,(\mathcal{P}) := \bigcup_{Q \in \mathcal{P}} Pred\,(Q).$$

Described verbally, $Pred\,(Q)$ is the set of all elements from V which are predecessors of Q (in E). $Pred\,(\mathcal{P})$ is the set of all predecessors of any $Q \in \mathcal{P}$.

We will compute the systems of pseudo-intents through maximal independent sets. Therefore, the following result is useful:

Lemma 2. *([23]) Let $\varnothing \neq \mathcal{P} \subseteq \mathbf{L}^M$. If $V \setminus \mathcal{P} = Pred\,(\mathcal{P})$, then $\mathcal{P}$ is a maximal independent set in $\mathbf{G}$.*

The next theorem characterises the systems of pseudo-intents of a fuzzy context using general hedges:

Theorem 5. *([23]) Let $\mathcal{P} \subseteq \mathbf{L}^M$. $\mathcal{P}$ is a system of pseudo-intents if and only if $V \setminus \mathcal{P} = Pred(\mathcal{P})$.*

It is well-known that the maximal independent sets of a graph can be efficiently enumerated in lexicographic order with only polynomial delay between the output of two successive independent sets ([24]). In [25] it was shown that the pseudo-intents cannot be enumerated in lexicographic order with polynomial delay unless P = NP. These two results do not contradict each other because they address different issues. The first one in encountered when we enumerate the maximal independent sets of the graph $\mathbf{G}$ which is the input of the corresponding algorithm. These sets correspond to the systems of pseudo-intents. Whereas the result from [25] is for the globalisation and takes as input a formal context enumerating its pseudo-intents.

In the following we will exemplify the computation of the systems of pseudo-intents. Afterwards, we illustrate how an attribute exploration with general hedge could be performed.

Example 4. We start with a very simple example. Let $(\{g\}, \{a, b\}, I)$ be the formal fuzzy context with $I(g, a) = 0.5$ and $I(g, b) = 0$. Further, we use the three-element Lukasiewicz chain with * being the identity. First, we compute V as given by (3) and obtain

$$V = \{\{^{0.5}/a, ^{0.5}/b\}, \{^{0.5}/b\}, \{\}, \{^{0.5}/a, b\}, \{b\}, \{a\}\}.$$

Afterwards, we compute the binary relation E as given by (4) which is displayed in Figure 6. Considering the undirected diagram of Figure 6 we obtain the graph $\mathbf{G}$. There, we have four maximal independent sets, namely

$$\mathcal{P}_1 = \{\{\}, \{^{0.5}/a, b\}, \{a\}\},$$
$$\mathcal{P}_2 = \{\{^{0.5}/b\}, \{a\}\},$$
$$\mathcal{P}_3 = \{\{b\}, \{a\}\},$$
$$\mathcal{P}_4 = \{\{^{0.5}/a, ^{0.5}/b\}, \{a\}\}.$$

$\mathcal{P}_1$ and $\mathcal{P}_3$ do not satisfy the condition of Theorem 5 and are therefore not

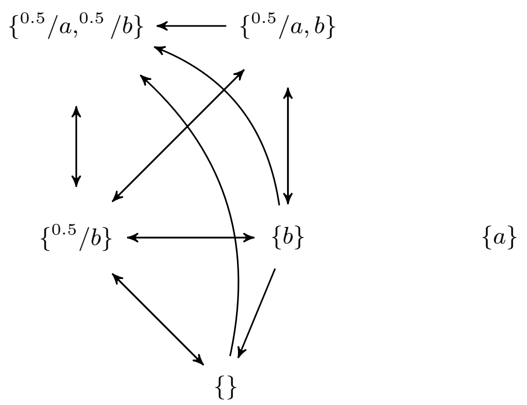

Fig. 6. Binary relation E for (G, M, I)

systems of pseudo-intents. $\mathcal{P}_2$ and $\mathcal{P}_4$ do satisfy this condition and hence they are systems of pseudo-intents yielding the stem bases displayed in Figure 7.

$\mathcal{T}_2$		$\mathcal{T}_4$	
(1)	$^{0.5}/b \Rightarrow a$	(3)	$^{0.5}/a, ^{0.5}/b \Rightarrow a$
(2)	$a \Rightarrow ^{0.5}/b$	(4)	$a \Rightarrow ^{0.5}/b$

Fig. 7. Stem bases

Now we could start an attribute exploration, for instance in $\mathcal{T}_2$. The algorithm would ask us:

Objects having attribute b to degree 0.5 *also have attribute a to degree* 1*?*

Let us answer this question affirmatively. The next question is:

Objects having attribute a to degree 1 *also have attribute b to degree* 0.5*?*

We deny this implication and provide a counterexample, namely the object h with $I(h, a) = 1$ and $I(h, b) = 0$. This counterexample obviously respects the already confirmed implication so the context is extended by the new object h. For this extended context we can compute the sets V and E. The binary relation

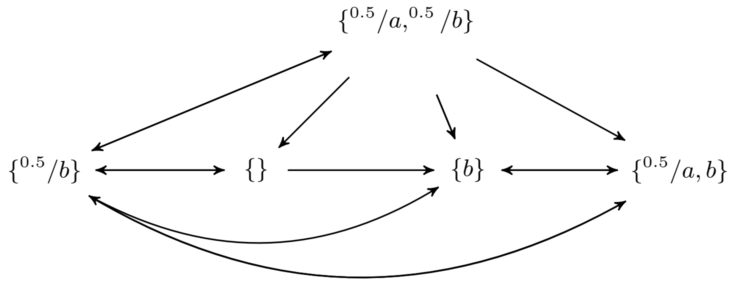

Fig. 8. Binary relation E for the extended context

E for the extended context is given in Figure 8. From this graph we obtain four maximal independent sets, three of which form systems of pseudo-intents. The stem bases which they induce are displayed in Figure 9. At the beginning we

$\mathcal{T}_2^{'}$	$\mathcal{T}_2^{''}$	$\mathcal{T}_2^{'''}$
(5) $^{0.5}/b \Rightarrow a$	(6) $\{\} \Rightarrow^{0.5}/a$ (7) $^{0.5}/a, b \Rightarrow b$	(8) $\{b\} \Rightarrow a$

Fig. 9. Stem bases of the extended context

have confirmed implication (1) from Figure 7. However, this implication is now not present any more in the stem bases $\mathcal{T}_2^{''}$ and $\mathcal{T}_2^{'''}$. This is also reflected in the stem base $\mathcal{T}_4$. Even though the counterexample respects implication (3), the pseudo-intent belonging to this implication also disappears.

Concluding, by extending the context with objects which respect the already confirmed implications, the latter may disappear from the stem base of the extended context. Hence, we do not have an analogon of Proposition 2 for general hedges.

The attribute exploration with general hedges raises a lot of questions and open problems. First of all it is unclear whether such an exploration is desirable. We have more than one stem base for a context. These bases are equally powerful with respect to their expressiveness. The major problem however is how to perform an attribute exploration successfully. It is an open problem how to enumerate the pseudo-intents obtained by general hedges such that the already confirmed implication still remain in the stem base of the extended context. One could for instance make some constraints on the counterexamples. However, such an approach is not in the spirit of attribute exploration.

5 Conclusion

We presented a generalisation of attribute exploration to the fuzzy setting. The problem is two-sided. If one uses the globalisation in the residuated lattice, the stem base is unique. For such a setting the results regarding the exploration from the crisp case can be transferred without problems and one can perform successfully an attribute exploration with attributes having fuzzy values. Using hedges different from the globalisation one obtains more than one system of pseudo-intents. This alone would not cause such a big problem. The major difficulty comes with the fact that the already confirmed pseudo-intents are not necessarily pseudo-intents of the extended context. This is therefore an open problem, how to perform an attribute exploration using a general hedge.

In the future we will focus on the problem regarding the general hedge and on extensions of this method, as for instance on fuzzy attribute exploration with background knowledge. There, the user can enter in advance some implications which he/she knows to hold between the attributes. Using such background knowledge one usually has to provide less examples and answer to fewer questions.

We are expecting that the method will have many practical applications, as its crisp variant has. Therefore, we will also focus on applications using attribute exploration in a fuzzy setting.

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